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Positive Solutions to Singular Sixth–Order Differential System

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In this paper, we investigate the existence of positive solutions for the singular sixth-order differential system with three variable parameters

$$u^{(4)} = \varphi u + f(t,u,\varphi), \quad 0 < t < 1$$

$$-\varphi^{(6)} + A(t)\varphi^{(4)} + B(t)\varphi'' + C(t)\varphi = \mu g(t,u,u''), \quad 0 < t < 1$$

$$u(0) = u(1) = u''(0) = u''(1) = 0,$$

$$\varphi(0) = \varphi(1) = \varphi''(0) = \varphi''(1) = \varphi^{(4)}(0) = \varphi^{(4)}(1) = 0,$$

where $\mu > 0$ is a constant, and the nonlinear terms f, g may be singular with respect to the time and space variables. Using a fixed point theorem in cones and an operator spectral theorem we give an new existence result for singular differential system. The existence of the positive solution depends on μ, i.e. there exists a positive number $\overline{\mu}$ such that if $0 < \mu < \overline{\mu}$, the boundary value problem has a positive solution.

Keywords: positive solutions; fixed point theorem; singular solutions; bending of an elastic beam; cone; boundary value problem; existence.

1. Introduction

Boundary value problems for ordinary differential equations can be used to describe a large number of chemical, biological and physical phenomena. The existence of positive solutions for such problems has become an important area of investigation in recent years. It is well known that the bending of an elastic beam can be described with fourth-order boundary value problems. An elastic beam with its two ends simply supported, can be described by the fourth-order boundary value problem

$$u^{(4)}(t) = f(t,u(t),u''(t)), \quad 0 < t < 1, \tag{1}$$

$$u(0) = u(1) = u''(0) = u''(1) = 0. \tag{2}$$

Existence of solutions for problem (1) was established for example by Gupta [1,2], Liu [3], Ma [4], Ma et. al. [5], Ma and Wang [6], Aftabizadeh [7], Yang [8], Del Pino and Manasevich [9], RP Agarwal et.al. [10,11,12] (see also the references therein). (see also the references therein). All of those results are based on the Leray-Schauder continuation method, topological degree and the method of lower and upper solutions.

Recently, Wang and An [13] studied the existence of positive solutions for a second-order differential system by using the fixed point theorem of cone expansion and compression.

It is well known that the deformation of the equilibrium state, an elastic circular ring segment can be described by a boundary value problem for a sixth-order ordinary differential equation. However, there are only a handful of articles on this topic.
In this paper we shall discuss the existence of positive solutions for the sixth-order boundary value problem

$$u^{(4)} = \varphi u + f(t,u,\varphi), \quad 0 < t < 1$$

$$-\varphi^{(6)} + A(t)\varphi^{(4)} + B(t)\varphi'' + C(t)\varphi = \mu g(t,u,u''), \quad 0 < t < 1$$

$$u(0) = u(1) = u''(0) = u''(1) = 0,$$

$$\varphi(0) = \varphi(1) = \varphi''(0) = \varphi''(1) = \varphi^{(4)}(0) = \varphi^{(4)}(1) = 0, \tag{3}$$

where μ is a positive parameter, $A(t), B(t), C(t), D(t) \in C[0,1]$, $D(t) > 0$ and $f(t,u,\varphi) : (0,1) \times [0,+\infty) \times [0+\infty) \longrightarrow (0,+\infty)$ and $g(t,u,v) : (0,1) \times (0,+\infty) \times (-\infty,0) \longrightarrow (0,+\infty)$ is continuous. In fact as we will see below one could consider in Section 2 and 3 $f(t,u,\varphi) = f_1(t) f_2(t,u,\varphi)$ with $f_2(t,u,\varphi) : [0,1] \times [0,+\infty) \times [0,+\infty) \longrightarrow (0,+\infty)$ and $f_1 : (0,1) \to (0,+\infty)$ is continuous, provided

$$\int_0^1 \int_0^1 K(\tau,\tau) K(\tau,s) f_1(s) ds\, d\tau < +\infty;$$

here K is as defined in Section 2. Moreover, our hypotheses allow but do not require $g(t,u,v) : [0,1] \times (0,+\infty) \times (-\infty,0) \to (0,+\infty)$ to be singular at $u = 0$, and at $v = 0$. The existence of the positive solution depends on μ, i.e. there exists a positive number $\overline{\mu}$ such that if $0 < \mu < \overline{\mu}$, the boundary value problem (3) has a positive solution. For this, we shall assume the following conditions throughout:

$(H1)$ $a = \sup_{t\in[0,1]} A(t) > -\pi^2$, $b = \inf_{t\in[0,1]} B(t) > 0$, $c = \sup_{t\in[0,1]} C(t) < 0$, $\pi^6 + a\pi^4 - b\pi^2 + c > 0$, where $a,b,c \in R$, $a = \lambda_1 + \lambda_2 + \lambda_3 > -\pi^2, b = -\lambda_1\lambda_2 - \lambda_2\lambda_3 - \lambda_1\lambda_3 > 0, c = \lambda_1\lambda_2\lambda_3 < 0$ and $\lambda_1 \geq 0 \geq \lambda_2 > -\pi^2, 0 \leq \lambda_3 < -\lambda_2$.

Assumption $(H1)$ involves a three-parameter nonresonance condition.

2. Preliminaries

Let $Y = C[0,1]$ and $Y_+ = \{u \in Y : u(t) \geq 0,\ t \in [0,1]\}$. It is well known that Y is a Banach space equipped with the norm $\|u\|_0 = \sup_{t\in[0,1]} |u(t)|$.

We denote the norm $\|u\|_2$ by

$$\|u\|_2 = \max\left\{\|u\|_0, \|u''\|_0\right\}.$$

It is easy to show that $Z = \left\{u \in C^2[0,1] : u(0) = u(1) = 0\right\}$ is complete with the norm $\|u\|_2$ and $\|u\|_2 \leq \|u\|_0 + \|u''\|_0 \leq 2\|u\|_2$.

Set $X=\left\{u\in C^4[0,1]:u(0)=u(1)=u''(0)=u''(1)=0\right\}$. For given $\chi\geq 0$ and $\nu\geq 0$, we denote the norm $\|\cdot\|_{\chi,\nu}$ by

$$\|\cdot\|_{\chi,\nu}=\sup_{t\in[0,1]}\left\{\left|u^{(4)}(t)\right|+\chi\left|u''(t)\right|+\nu\left|u(t)\right|\right\},\quad u\in X.$$

We also need the space X equipped with the norm

$$\|u\|_4=\max\left\{\|u\|_0,\|u''\|_0,\left\|u^{(4)}\right\|_0\right\}.$$

In [11], it is shown that X is complete with the norms $\|\cdot\|_{\chi,\nu}$ and $\|u\|_4$, and moreover $\forall u\in X,\ \|u\|_0\leq\|u''\|_0\leq\left\|u^{(4)}\right\|_0$.

We will investigate the existence of positive solutions for problem (3) by the following fixed point theorem of cone expansion and compression of norm type:

Lemma 1 ([14]). Let E be a real Banach space and let $P\subset E$ be a cone in E. Assume Ω_1, Ω_2 are open subset of E with $\theta\in\Omega_1$, $\overline{\Omega}_1\subset\Omega_2$, and let $T:P\cap\left(\overline{\Omega}_2\backslash\Omega_1\right)\to P$ be a completely continuous operator such that either

(i) $\|Tu\|\leq\|u\|$, $u\in P\cap\partial\Omega_1$ and $\|Tu\|\geq\|u\|$, $u\in P\cap\partial\Omega_2$; or

(ii) $\|Tu\|\geq\|u\|$, $u\in P\cap\partial\Omega_1$ and $\|Tu\|\leq\|u\|$, $u\in P\cap\partial\Omega_2$.

Then T has a fixed point in $P\cap\left(\overline{\Omega}_2\backslash\Omega_1\right)$.

Firstly, we will transform the problem (3) into a new form.

For $h\in Y$, consider the following linear boundary value problem:

$$-\varphi^{(6)}+a\varphi^{(4)}+b\varphi''+c\varphi=h(t),\quad 0<t<1$$

$$\varphi(0)=\varphi(1)=\varphi''(0)=\varphi''(1)=\varphi^{(4)}(0)=\varphi^{(4)}(1)=0,\tag{4}$$

where a,b,c satisfy the assumption

$$\pi^6+a\pi^4-b\pi^2+c>0\tag{5}$$

and let $\Gamma=\pi^6+a\pi^4-b\pi^2+c$. The inequality (5) follows immediately from the fact that $\Gamma=\pi^6+a\pi^4-b\pi^2+c$ is the first eigenvalue of the problem $-\varphi^{(6)}+a\varphi^{(4)}+b\varphi''+c\varphi=\lambda\varphi$, $\varphi(0)=\varphi(1)=\varphi''(0)=\varphi''(1)=\varphi^{(4)}(0)=\varphi^{(4)}(1)=0$ and $\phi_1(t)=\sin\pi t$ is the first eigenfunction, i.e. $\Gamma>0$. Because the line $l_1=\left\{(a,b,c):\pi^6+a\pi^4-b\pi^2+c=0\right\}$ is the first eigenvalue line of the three-parameter boundary value problem $-\varphi^{(6)}+a\varphi^{(4)}+b\varphi''+c\varphi=0$, $\varphi(0)=\varphi(1)=\varphi''(0)=\varphi''(1)=\varphi^{(4)}(0)=\varphi^{(4)}(1)=0$, if (a,b,c) lies in l_1, then by the Fredholm alternative the existence of a solution of the boundary value problem (4) cannot be guaranteed.

Let $P(\lambda)=\lambda^2+\beta\lambda-\alpha$ where $\beta<2\pi^2,\alpha\geq 0$. It is easy to see that equation $P(\lambda)=0$ has two real roots $\lambda_1,\lambda_2=\frac{-\beta\pm\sqrt{\beta^2+4\alpha}}{2}$, with $\lambda_1\geq 0\geq\lambda_2>-\pi^2$. Let λ_3 be a number such that $0\leq\lambda_3<-\lambda_2$. In this case, (4) satisfies the following decomposition form:

$$-\varphi^{(6)}+a\varphi^{(4)}+b\varphi''+c\varphi=(-\frac{d^2}{dt^2}+\lambda_1)(-\frac{d^2}{dt^2}+\lambda_2)(-\frac{d^2}{dt^2}+\lambda_3)\varphi,\quad 0<t<1.\tag{6}$$

It is obvious that $a=\lambda_1+\lambda_2+\lambda_3>-\pi^2, b=-\lambda_1\lambda_2-\lambda_2\lambda_3-\lambda_1\lambda_3>0, c=\lambda_1\lambda_2\lambda_3<0$.

Suppose that $G_i(t,s)(i=1,2,3)$ is the Green function associated with

$$-\varphi''+\lambda_i\varphi=0,\quad u(0)=u(1)=0. \tag{7}$$

We need the following lemmas.

Lemma 2 ([14]). Let $\omega_i=\sqrt{|\lambda_i|}$, then $G_i(t,s)(i=1,2,3)$ can be expressed as
(i) when $\lambda_i>0, G_i(t,s)=\left\{\begin{array}{l}\dfrac{\sinh\omega_i t\sinh\omega_i(1-s)}{\omega_i\sinh\omega_i},\ 0\le t\le s\le 1\\ \dfrac{\sinh\omega_i s\sinh\omega_i(1-t)}{\omega_i\sinh\omega_i},\ 0\le s\le t\le 1\end{array}\right\}$
(ii) when $\lambda_i=0, K(t,s)=G_i(t,s)=\left\{\begin{array}{l}t(1-s),\ 0\le t\le s\le 1\\ s(1-t),\ 0\le s\le t\le 1\end{array}\right\}$
(iii)when $-\pi^2<\lambda_i<0, G(t,s)=\left\{\begin{array}{l}\dfrac{\sin\omega_i t\sin\omega_i(1-s)}{\omega_i\sin\omega_i},\ 0\le t\le s\le 1\\ \dfrac{\sin\omega_i s\sin\omega_i(1-t)}{\omega_i\sin\omega_i},\ 0\le s\le t\le 1\end{array}\right\}$.

Lemma 3 ([14]). $G_i(t,s)(i=1,2,3)$ has the following properties:
(i) $G_i(t,s)>0,\ \forall t,s\in(0,1)$;
(ii) $G_i(t,s)\le C_iG_i(s,s),\ \forall t,s\in[0,1]$;
(iii) $G_i(t,s)\ge\delta_iG_i(t,t)G_i(s,s),\ \forall t,s\in[0,1]$;
(iv) $|K(t_1,s)-K(t_2,s)|\le 2|t_1-t_2|,\ \forall t_1,t_2,s\in[0,1]$;
where $C_i=1,\delta_i=\frac{\omega_i}{\sinh\omega_i}$, if $\lambda_i>0; C_i=1,\delta_i=1$, if $\lambda_i=0; C_i=\frac{1}{\sin\omega_i},\delta_i=\omega_i\sin\omega_i$, if $-\pi^2<\lambda_i<0$.

Proof. It can be easily seen that (i), (ii) and (iii) are satisfied. Next, we check that (iv) is satisfied. In fact, for $t_1\le t_2\le s$, or $s\le t_1\le t_2$, it is easy to know that $|K(t_1,s)-K(t_2,s)|\le|t_1-t_2|$.

Similarly, for $t_1\le s\le t_2$, we have

$$|K(t_1,s)-K(t_2,s)|\le|s(1-t_2)-t_1(1-s)|\le|s(t_1-t_2)+s-t_1|\le 2|t_1-t_2|.$$

This proves that (iv) is satisfied.

This finishes the proof. □

In what follows, we shall let $D_i=\int_0^1 G_i(s,s)ds$.

Now, since

$$-\varphi^{(6)}+a\varphi^{(4)}+b\varphi''+c\varphi=(-\frac{d^2}{dt^2}+\lambda_1)(-\frac{d^2}{dt^2}+\lambda_2)(-\frac{d^2}{dt^2}+\lambda_3)\varphi$$

$$=(-\frac{d^2}{dt^2}+\lambda_2)(-\frac{d^2}{dt^2}+\lambda_1)(-\frac{d^2}{dt^2}+\lambda_3)\varphi=h(t), \tag{8}$$

the solution of boundary value problem (4) can be expressed by

$$\varphi(t)=\int_0^1\int_0^1\int_0^1 G_1(t,v)G_2(v,s)G_3(s,\tau)h(\tau)d\tau dsdv,\quad t\in[0,1]. \tag{9}$$

Thus, for every given $h\in Y$, the boundary value problem (4) has a unique solution $\varphi\in C^6[0,1]$ which is given by (9).

We now define a mapping $Q : C[0,1] \to C[0,1]$ by

$$(Qh)(t) = \int_0^1 \int_0^1 \int_0^1 G_1(t,v)G_2(v,s)G_3(s,\tau)h(\tau)d\tau ds dv, \quad t \in [0,1]. \tag{10}$$

Throughout this article we shall denote $Qh = \varphi$ the unique solution of the linear boundary value problem (4).

Let us introduce the following notation:

$$\widehat{Q}h = \int_0^1 \int_0^1 \int_0^1 C_1 G_1(v,v)G_2(v,s)G_3(s,\tau)h(\tau)d\tau ds dv. \tag{11}$$

Lemma 4. $Q : Y \to (X, \|\cdot\|_{\chi,\nu})$ is linear and completely continuous where $\chi = \lambda_1 + \lambda_3, \nu = \lambda_1\lambda_3$ and $\|Q\| \le D_2$.

Proof. The proof of completely continuous is similar to the proof of Lemma 6 in [15], so we omit it. Next we will show that $\|Q\| \le D_2$. Assume that $h \in Y$ and $\varphi = Qh$ is the solution the boundary value problem (4). It is clear that the operator Q maps Y into X. Now for all $\forall h \in Y, \varphi = Qh \in X$, $\varphi(0) = \varphi(1) = \varphi''(0) = \varphi''(1) = \varphi^{(4)}(0) = \varphi^{(4)}(1) = 0$. Using (8) it is easy to see that

$$-\varphi'' + \lambda_i \varphi = \int_0^1 \int_0^1 G_j(t,v)G_k(v,\tau)h(\tau)d\tau dv, \quad t \in [0,1]. \tag{12}$$

and

$$\varphi^{(4)} - (\lambda_i + \lambda_j)\varphi'' + \lambda_i\lambda_j\varphi = \int_0^1 G_k(t,v)h(v)dv, \quad t \in [0,1]. \tag{13}$$

where $i, j, k = 1,2,3$ and $i \ne j \ne k$.

We will now show $\|Qh\|_{\chi,\nu} \le D_2 \|h\|_0, \forall h \in Y,$ where $\chi = \lambda_1 + \lambda_3 \ge 0, \nu = \lambda_1\lambda_3 \ge 0$. For this, $\forall h \in Y_+$, let $\varphi = Qh$, and by Lemma 3, $\varphi \in X \cap Y_+$. The equality (12) with the assumption $\lambda_2 \le 0$ implies that $\varphi'' \le 0$. Similarly, the equality (13) with the assumptions $\lambda_2 + \lambda_3 < 0$ and $\lambda_2\lambda_3 \le 0$ implies that $\varphi^{(4)} \ge 0$.

From (13) with $\chi = \lambda_1 + \lambda_3 \ge 0$, $\nu = \lambda_1\lambda_3 \ge 0$ and $\varphi \ge 0$, $\varphi'' \le 0$, $\varphi^{(4)} \ge 0$ we immediately have

$$\left|\varphi^{(4)}(t)\right| + \chi\left|\varphi''(t)\right| + \nu\left|\varphi(t)\right| = \varphi^{(4)} - (\lambda_1 + \lambda_3)\varphi'' + \lambda_1\lambda_3\varphi = \int_0^1 G_2(t,v)h(v)dv, \quad t \in [0,1]. \tag{14}$$

For any $h \in Y$, let $h = h_1 - h_2, \varphi_1 = Th_1, \varphi_2 = Th_2$, where h_1, h_2 are the positive part and negative part of h, respectively. Let $\varphi = Th$, then $\varphi = \varphi_1 - \varphi_2$. From the above, we have $\varphi_i \ge 0, \varphi_i'' \le 0, \varphi_i^{(4)} \ge 0, i = 1,2,$ and the following equality holds:

$$\left|\varphi_i^{(4)}(t)\right| + (\lambda_1 + \lambda_3)\left|\varphi_i''(t)\right| + \lambda_1\lambda_3\left|\varphi_i(t)\right| = \int_0^1 G_2(t,v)h_i(v)dv = \widehat{H}h_i, \quad t \in [0,1], \quad i = 1,2. \tag{15}$$

So, from (15), we have

$$\left|\varphi^{(4)}(t)\right| + (\lambda_1 + \lambda_3)\left|\varphi''(t)\right| + \lambda_1\lambda_3\left|\varphi(t)\right| = \left|\varphi_1^{(4)}(t) - \varphi_2^{(4)}(t)\right|$$

$$+(\lambda_1 + \lambda_3)\left|\varphi_1''(t) - \varphi_2''(t)\right| + \lambda_1\lambda_3\left|\varphi_1(t) - \varphi_2(t)\right|$$

$$\le \left(\left|\varphi_1^{(4)}(t)\right| + (\lambda_1 + \lambda_3)\left|\varphi_1''(t)\right| + \lambda_1\lambda_3\left|\varphi_1(t)\right|\right)$$

$$+\left(\left|\varphi_2^{(4)}(t)\right|+(\lambda_1+\lambda_3)\left|\varphi_2''(t)\right|+\lambda_1\lambda_3\left|\varphi_2(t)\right|\right)$$

$$=\widehat{H}h_1+\widehat{H}h_2=\widehat{H}\,|h|\leq D_2\,\||h|\|_0=D_2\,\|h\|_0\,.$$

Thus $\|Qh\|_{\chi,\nu}\leq D_2\,\|h\|_0$, and hence $\|Q\|\leq D_2$. ∎

We consider the existence of a positive solution of the second equation of (3) (the function $\varphi\in C^6(0,1)\cap C^4[0,1]$ is a positive solution of the second equation of (3), if $\varphi(t)\geq 0$, $t\in[0,1]$, and $\varphi\neq 0$). It is easy to see that the second equation of (3) is equivalent to the following boundary value problem:

$$-\varphi^{(6)}+a\varphi^{(4)}+b\varphi''+c\varphi=-\left(A(t)-a\right)\varphi^{(4)}-\left(B(t)-b\right)\varphi''-\left(C(t)-c\right)\varphi$$

$$+\mu g(t,u,u''). \tag{16}$$

For any $\varphi\in X$, let

$$|(G\varphi)(t)|=-\left(A(t)-a\right)\varphi^{(4)}-\left(B(t)-b\right)\varphi''-\left(C(t)-c\right)\varphi.$$

The operator $G:X\to Y$ is linear. By Lemma 4 and Corollary 10, $\forall\varphi\in X$, $t\in[0,1]$, we have

$$|(G\varphi)(t)|\leq\left[-A(t)+B(t)-C(t)-(-a-b-c)\right]\|\varphi\|_4$$

$$\leq K\|\varphi\|_4\leq K\|\varphi\|_{\chi,\nu}$$

where $K=\max_{t\in[0,1]}\left[-A(t)+B(t)-C(t)-(-a+b-c)\right]$, $\chi=\lambda_2+\lambda_3\geq 0$, $\nu=\lambda_2\lambda_3\geq 0$. Hence $\|G\varphi\|_0\leq K\,\|\varphi\|_{\chi,\nu}$, and so $\|G\|\leq K$. Also $\varphi\in C^4[0,1]\cap C^6(0,1)$ is a solution of (16) iff $\varphi\in X$ satisfies $\varphi=Q\left(G\varphi+h_1\right)$, where $h_1(t)=\mu g(t,u,u'')$ i.e.

$$\varphi\in X,\quad (I-QG)\,\varphi=Qh_1. \tag{17}$$

The operator $I-QG$ maps X into X. From $\|Q\|\leq D_2$ together with $\|G\|\leq K$ and condition $D_2K<1$, and applying the operator spectra theorem, we find that $(I-QG)^{-1}$ exists and bounded. Let $L=D_2K$.

Let $H=(I-QG)^{-1}Q$. Then (17) is equivalent to $\varphi=Hh_1$. By the Neumann expansion formula, H can be expressed by

$$H=\left(I+QG+...+(QG)^n+...\right)\,Q=Q+(QG)Q+...+(QG)^n\,Q+.... \tag{18}$$

The complete continuity of Q with the continuity of $(I-QG)^{-1}$ guarantees that the operator $H:Y\to X$ is completely continuous.

Now $\forall h\in Y_+$, let $\varphi=Qh$, then $\varphi\in X\cap Y_+$, and $\varphi''\leq 0$, $\varphi^{(4)}\geq 0$.

Thus we have

$$(G\varphi)(t)=-\left(A(t)-a\right)\varphi^{(4)}-\left(B(t)-b\right)\varphi''-\left(C(t)-c\right)\varphi\geq 0,\ \ t\in[0,1]\,.$$

Hence

$$\forall h\in Y_+,\quad (GQh)\,(t)\geq 0,\quad t\in[0,1] \tag{19}$$

and so $(QG)(Qh)(t) = Q(GQh)(t) \geq 0,\ \ t \in [0,1]$.

It is easy to see [15] that the following inequalities hold: $\forall h \in Y_+$,

$$\frac{1}{1-L}(Qh)(t) \geq (Hh)(t) \geq (Qh)(t), \quad t \in [0,1], \tag{20}$$

moreover,

$$\|(Hh)\|_0 \leq \frac{1}{1-L}\|(Qh)\|_0. \tag{21}$$

For any $u \in Y_+$, it is easy to see that $\varphi \in C^4[0,1] \cap C^6(0,1)$ being a positive solution of the second equation of (3) is equivalent to $\varphi \in Y_+$ being a nonzero solution of

$$\varphi(t) = \mu Hg(s,u(s),u''(s))(t). \tag{22}$$

Obviously, $H : Y_+ \to Y_+$ is completely continuous.

Thus inserting (22) into the first equation of (3), we have

$$u^{(4)}(t) = \mu u(t)Hg(s,u(s),u''(s))(t) + f(t,u(t),\mu Hg(s,u(s),u''(s))(t)),$$

$$u(0) = u(1) = u''(0) = u''(1) = 0. \tag{23}$$

Now we consider the existence of a positive solution of (23). The function $u \in C^4(0,1) \cap C^2[0,1]$ is a positive solution of (23), if $u(t) \geq 0$, $t \in [0,1]$, and $u \neq 0$.

Then the solution of (23) can be expressed as

$$u(t) = \mu \int_0^1 \int_0^1 K(t,\tau)K(\tau,s)u(s)Hg(v,u(v),u''(v))(s)\,ds\,d\tau +$$

$$+ \int_0^1 \int_0^1 K(t,\tau)K(\tau,s)f(s,u(s),\mu Hg(v,u(v),u''(v))(s))\,ds\,d\tau. \tag{24}$$

We recall that $Z = \left\{u \in C^2[0,1] : u(0) = u(1) = 0\right\}$ is complete with the norm $\|u\|_2 = \max\{\|u\|_0, \|u''\|_0\}$ and using Lemma 8 and Corollary 9, we have $\|u\|_2 = \|u''\|_0$. Throughout this paper, we use the Banach space $(Z, \|u''\|_0)$ to solve the problem (23).

Set

$$P = \left\{u \in Z, u(t) \geq K(t,t)\|u\|_0,\ -u''(t) \geq K(t,t)\left\|u''\right\|_0,\ t \in [0,1]\right\},$$

where $K(t,t) = t(1-t),\ \ t \in [0,1]$.

Note, P is a cone in Z. For $R > 0$, write $B_R = \{u \in C^2[0,1] : \|u\|_2 < R\}$.

It is easy to see that if $u \in P$ than

$$-u''(t) \geq \sigma \left\|u''\right\|_0, \ \ t \in \left[\frac{1}{4}, \frac{3}{4}\right], \tag{25}$$

where $\sigma = \frac{3}{16}$.

We now define a mapping $T : P \to C[0,1]$ by

$$Tu(t) = \mu \int_0^1 \int_0^1 K(t,\tau) K(\tau,s) u(s) Hg(t,u,u'') \, ds \, d\tau +$$

$$+ \int_0^1 \int_0^1 K(t,\tau) K(\tau,s) f(s,u(s), \mu Hg(t,u,u'')) \, ds \, d\tau. \tag{26}$$

Lemma 5. Let $u \in P$. Then the following relations hold:

(a) $(Tu)(t) \geq K(t,t) \|Tu\|_0$ for $t \in [0,1]$, and

(b) $-(Tu)''(t) \geq K(t,t) \|Tu''\|_0$ for $t \in [0,1]$.

Proof. For simplicity we denote

$$I = \mu \int_0^1 \int_0^1 K(\tau,\tau) K(\tau,s) u(s) q(s) ds d\tau +$$

$$+ \int_0^1 \int_0^1 K(\tau,\tau) K(\tau,s) h(s) ds d\tau,$$

$$J = \mu \int_0^1 K(s,s) u(s) q(s) ds + \int_0^1 K(s,s) h(s) ds,$$

and

$$q(s) = Hg(t,u,u'')(s), \ \ h(s) = f(s,u(s), \mu Hg(t,u,u'')(s)).$$

From Lemma 3 it is easy to see that

$$K(t,t) I \leq Tu(t) \leq I \ \text{ and } t \in [0,1] \tag{27}$$

$$K(t,t) J \leq -(Tu)''(t) \leq J, \qquad t \in [0,1] \tag{28}$$

Using (27-28), we have

$$\|Tu\|_0 \leq I, \ \text{ and } \left\|-(Tu)''\right\|_0 \leq J,$$

hence

$$(Tu)(t) \geq K(t,t)\,\|Tu\|_0 \text{ for } t \in [0,1] \text{ and}$$

$$-(Tu)''(t) \geq K(t,t)\,\|Tu''\|_0 \text{ for } t \in [0,1].$$

This finishes the proof. □

Throughout this paper, we assume additionaly that the continuous function $f(t,u,\varphi) : (0,1) \times [0,+\infty) \times [0+\infty) \longrightarrow (0,+\infty)$ satisfies

$(H2)$

$$f(t,u,v) \leq f_1(t) f_2(uv), \quad t \in (0,1), \quad u,v \in R^+,$$

where $f_1 : (0,1) \to (0,+\infty)$ and $f_2 : [0,+\infty) \to (0,+\infty)$ is continuous, $R^+ = [0,+\infty)$, $R^- = (-\infty,0]$.

Moreover the function $g(t,u,v) : [0,1] \times (0,+\infty) \times (-\infty,0) \to [0,+\infty)$ satisfies

$(H3)$ There exists an $a > 0$ such that $g(t,u,v)$ is nonincreasing in $u \leq a$ and $|v| \leq a$ for each fixed $t \in [0,1]$ i.e. if $-a \leq v_2 \leq v_1 < 0$ and $0 < u_1 \leq u_2$ then $g(t,u_1,v_1) \geq g(t,u_2,v_2)$.

$(H4)$ There exists an function $g_1(t,w) : [0,1] \times (0,+\infty) \to [0,+\infty)$ such that $g_1(t,w)$ is nonincreasing in $u \leq a$ for each fixed $t \in [0,1]$, i.e. if $0 < w_1 \leq w_2$ then $g(t,w_1) \geq g(t,w_2)$

and each fixed $0 < r \leq a$

$$0 < \int_0^1 g_1(s, rs(1-s))ds < \infty.$$

So, we assume additionaly that the function $g(t,u,v)$ satisfies

$$g(t,u,v) \leq g_1(t, u+|v|), \quad t \in [0,1], \quad u \in (0,+\infty), \quad v \in (-\infty,0).$$

Let us introduce the following notations:

$$D_1 = \int_0^1 \int_0^1 K(\tau,\tau) K(\tau,s)\, ds\, d\tau, \quad D_4 = \int_0^1 K(s,s)\, f_1(s)\, ds,$$

$$D_2 = \int_0^1 \int_0^1 K(\tau,\tau) K(\tau,s) f_1(s)\, ds\, d\tau, \quad D_3 = \int_0^1 K(\tau,\tau) d\tau,$$

$$D_5 = \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} K(\frac{1}{2},\tau) K(\tau,s) G_1(s,v) G_2(v,z) G_3(z,x) dx dz dv\, ds\, d\tau,$$

$$D_6=\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}}G_1(v,v)G_2(v,z)G_3(z,x)dxdzdv.$$

Lemma 6. Let $(H1),(H2),(H3)$ and $(H4)$ hold. Then for all $u\in P\cap\overline{B}_R/B_r$ where $r<a<R$ the following hold

$$(Tu)(t)\leq\frac{\mu}{1-L}D_3\|u\|_0M_r+D_4\sup_{s\in(0,1)}f_2(\frac{\mu}{1-L}u(s)Hg(v,u,u'')(s)),$$

and

$$-(Tu)''(t)\leq\frac{\mu}{1-L}D_3\|u\|_0M_r+D_4\sup_{s\in(0,1)}f_2(\frac{\mu}{1-L}u(s)Hg(v,u,u'')(s)),$$

where

$$M_r=\int_0^1\int_0^1\int_0^1\cdot C_1G_1(w,w)G_2(w,z)G_3(z,v)g_1(v,rK(v,v))dvdzdw+$$

$$+\sup_{x\in(0,R]}\sup_{p\in[r,R]}\int_0^1\int_0^1\int_0^1C_1G_1(w,w)G_2(w,z)G_3(z,v)g(v,x,p)\,dvdzdw.$$

Proof. It is easy to see that $D_1\leq D_3$, and $D_2\leq D_4$. Let $u\in P\cap\overline{B}_R/B_r$, then by Lemma 8, $\|u\|_0\leq\|u''\|_0$ and by Corollary 9, $\|u\|_2=\|u''\|_0$. Thus $r\leq\|u''\|_0\leq R$. Also since $u\in P$ we have $-u''(t)\geq K(t,t)\|u''\|_0$, $u(t)\geq K(t,t)\|u\|_0$, and $u(t)+|u''(t)|\geq rK(t,t)$, $t\in[0,1]$.

By Lemma 1. and $(H3)-(H5)$ we have

$$Tu(t)=\mu\int_0^1\int_0^1K(t,\tau)K(\tau,s)\,u(s)Hg(v,u,u'')(s)\,ds\,d\tau+$$

$$+\int_0^1\int_0^1K(t,\tau)K(\tau,s)\,f(s,u(s),\mu Hg(v,u,u'')(s))\,ds\,d\tau$$

$$\leq\frac{\mu}{1-L}\int_0^1\int_0^1K(t,\tau)\,K(\tau,s)\,u(s)Qg(v,u(v),u'')(s))\,dv\,ds\,d\tau+$$

$$+\int_0^1\int_0^1K(t,\tau)K(\tau,s)\,f(s,u(s),\mu Hg(v,u(v),u'')(s))\,ds\,d\tau=$$

$$=\frac{\mu}{1-L}\int_0^1\int_0^1K(t,\tau)\,K(\tau,s)\,u(s)\int_0^1\int_0^1\int_{u(v)+|u''(v)|\leq a}\cdot$$

$$\cdot G_1(s,w)G_2(w,z)G_3(z,v)g(v,u(v),u''(v))\,dvdzdw\,ds\,d\tau +$$

$$+\frac{\mu}{1-L}\int_0^1\int_0^1 K(t,\tau)\,K(\tau,s)\,u(s)\int_0^1\int_0^1 \int\limits_{u(v)+|u''(v)|\geq a} \cdot$$

$$\cdot G_1(s,w)G_2(w,z)G_3(z,v)g(v,u(v),u''(v))\,dvdzdw\,ds\,d\tau +$$

$$+\int_0^1\int_0^1 K(t,\tau)K(\tau,s)\,f(s,u(s),\mu Hg(v,u(v),u'')(s))\,ds\,d\tau$$

$$\leq \frac{\mu}{1-L}\int_0^1\int_0^1 K(t,\tau)\,K(\tau,s)\,\|u\|_0\int_0^1\int_0^1 \int\limits_{u(v)+|u''(v)|\leq a} \cdot$$

$$\cdot C_1 G_1(w,w)G_2(w,z)G_3(z,v)g_1(v,rK(v,v))dvdzdw\,ds\,d\tau +$$

$$+\frac{\mu}{1-L}\int_0^1\int_0^1 K(t,\tau)\,K(\tau,s)\,\|u\|_0\int_0^1\int_0^1 \sup_{x\in(0,R]}\ \sup_{p\in[r,R]} \int\limits_{u(v)+|u''(v)|\geq a} \cdot$$

$$\cdot G_1(s,w)G_2(w,z)G_3(z,v)g(v,x,p)\,dvdzdw\,ds\,d\tau +$$

$$+\int_0^1\int_0^1 K(t,\tau)K(\tau,s)\,f_1(s),f_2(\mu u(s)Hg(v,u(v),u'')(s))\,ds\,d\tau$$

$$\leq \frac{\mu}{1-L}\int_0^1\int_0^1 K(t,\tau)\,K(\tau,s)\,\|u\|_0\int_0^1\int_0^1 \int\limits_{u(v)+|u''(v)|\leq a} \cdot$$

$$\cdot C_1 G_1(w,w)G_2(w,z)G_3(z,v)g_1(v,rK(v,v))dvdzdw\,ds\,d\tau +$$

$$+\frac{\mu}{1-L}\int_0^1\int_0^1 K(t,\tau)\,K(\tau,s)\,\|u\|_0\int_0^1\int_0^1 \sup_{x\in(0,R]}\ \sup_{p\in[r,R]} \int\limits_{u(v)+|u''(v)|\geq a} \cdot$$

$$\cdot G_1(s,w)G_2(w,z)G_3(z,v)g(v,x,p)\,dvdzdw\,ds\,d\tau +$$

$$+\int_0^1\int_0^1 K(t,\tau)K(\tau,s)\,f_1(s),\,ds\,d\tau \sup_{s\in(0,1)} f_2(\frac{\mu}{1-L}u(s)Hg(v,u(v),u'')(s))$$

$$\leq \frac{\mu}{1-L}\int_0^1\int_0^1 K(\tau,\tau)\,K(\tau,s)\,\|u\|_0\int_0^1\int_0^1\int_0^1 \cdot$$

$$\cdot C_1G_1(w,w)G_2(w,z)G_3(z,v)g_1(v,rK(v,v))dvdzdw\,ds\,d\tau+$$

$$+\frac{\mu}{1-L}\int_0^1\int_0^1 K(\tau,\tau)\,K(\tau,s)\,\|u\|_0\int_0^1\int_0^1 \sup_{x\in(0,R]}\sup_{p\in[r,R]}\int_0^1 \cdot$$

$$\cdot C_1G_1(w,w)G_2(w,z)G_3(z,v)g(v,x,p)\,dvdzdw\,ds\,d\tau+$$

$$+\int_0^1\int_0^1 K(\tau,\tau)K(\tau,s)\,f_1(s),\,ds\,d\tau \sup_{s\in(0,1)} f_2(\frac{\mu}{1-L}u(s)Hg(v,u(v),u'')(s))$$

$$\leq \frac{\mu}{1-L}D_1M_r\,\|u\|_0+D_2\sup_{s\in(0,1)} f_2(\frac{\mu}{1-L}u(s)Hg(v,u(v),u'')(s))$$

$$\leq \frac{\mu}{1-L}D_3M_r\,\|u\|_0+D_4\sup_{s\in(0,1)} f_2(\frac{\mu}{1-L}u(s)Hg(v,u(v),u'')(s)),$$

and similarly we also have

$$-(Tu)''(t)\leq \frac{\mu}{1-L}\int_0^1\int_0^1 K(\tau,\tau)\,K(\tau,s)ds\,d\tau\,\|u\|_0M_r+$$

$$+\int_0^1 K(s,s)\,f_1(s)\,ds\,d\tau \sup_{s\in(0,1)} f_2(\frac{\mu}{1-L}u(s)Hg(v,u(v),u'')(s))$$

$$\leq \frac{\mu}{1-L}D_3\,\|u\|_0M_r+D_4\sup_{s\in(0,1)} f_2(\frac{\mu}{1-L}u(s)Hg(v,u(v),u'')(s)).$$

This finishes the proof. □

Lemma 7. $T(P)\subset P$ and $T: P\cap(\overline{B}_R/B_r)\to P$ is completely continuous.

Proof. First, we prove that $T(P) \subset P$. To do this, let $u \in P$, then we define mapping $T : P \to C^2[0,1]$ by (26). Then for any $u \in P$, it is clear that

$$(Tu)''(t) = -\mu \int_0^1 K(t,s)\, u(s) Hg(v,u,u'')(s)\, ds$$

$$-\int_0^1 K(t,s)\, f(s,u(s),\mu Hg(v,u,u'')(s))\, ds \ \leq 0. \tag{29}$$

By Lemma 3,

$$Tu(t) \geq K(t,t)\, \|Tu\|_0, \quad t \in [0,1]$$

and

$$-(Tu)''(t) \geq K(t,t)\, \left\|(Tu)''\right\|_0 \quad t \in [0,1].$$

Hence $T(P) \subset P$.

We recall that

$$\widehat{Q}h = \int_0^1 \int_0^1 \int_0^1 C_1 G_1(v,v) G_2(v,s) G_3(s,\tau) h(\tau) d\tau ds dv. \tag{30}$$

Let us introduce the following notation:

$$N_r = \widehat{Q}g_1(\tau, rK(\tau,\tau)) = \int_0^1 \int_0^1 \int_0^1 C_1 G_1(v,v) G_2(v,s) G_3(s,\tau) g_1(\tau, rK(\tau,\tau))) d\tau ds dv. \tag{31}$$

Let $V \subset P \cap (\overline{B}_R / B_r)$ be a bounded set. Then there exists a $d > 0$, such that $\sup\{\|u\|_2 : u \in V\} = d$.

First we prove $T(V)$ is bounded. Since $\|u\|_2 = \max\{\|u\|_0, \|u''\|_0\}$, we have $u(t) + |u''(t)| \leq \|u\|_0 + \|u''\|_0 \leq 2d$, and $|\mu u(t) Hg(v,u(v),u''(v))| \leq \frac{\mu}{1-L} \|u\|_0 \widehat{Q}g_1(v, rK(v,v)) = \frac{\mu}{1-L} dN_r$ for all $t \in [0,1]$. Let $M_d = \sup\{f_2(w) : w \in [0, \frac{\mu}{1-L} dN_r]\}$. Now, from Lemma 3 and Lemma 6, we have for any $u \in V$ and $t \in [0,1]$ that

$$|Tu(t)| = |\, \mu \int_0^1 \int_0^1 K(t,\tau)\, K(\tau,s)\, u(s) Hg(v,u(v),u''(v))\, dv\, ds\, d\tau$$

$$+ \int_0^1 \int_0^1 K(t,\tau) K(\tau,s)\, f(s,u(s),\mu Hg(v,u(v),u''(v)))\, ds\, d\tau\, |$$

$$\leq \frac{\mu}{1-L} D_3 \|u\|_0 M_r + D_4 \sup_{s \in (0,1)} f_2(\mu u(s) Hg(v,u(v),u''(v)))$$

$$\leq \frac{\mu D_3 d M_r}{1-L} + D_4 \sup_{s\in(0,1)} f_2(\mu u(s) Hg(v,u(v),u''(v))) \leq \frac{\mu D_3 d M_r}{1-L} + M_d D_4. \tag{32}$$

We have a similar type inequality for $|(Tu)''(t)|$. Therefore $T(V)$ is bounded.

Next, we prove that $T(V)$ is equicontinuous. Now, from Lemma 3 and Lemma 6, we have for any $u \in V$ and any $t_1, t_2 \in [0,1]$ that

$$|(Tu)(t_1) - (Tu)(t_2)|$$

$$\leq \mu \int_0^1 \int_0^1 |K(t_1,\tau) - K(t_2,\tau)|\, K(\tau,s)\, u(s) Hg(v,u(v),u'')(v)(s)\, ds\, d\tau +$$

$$+ \int_0^1 \int_0^1 |K(t_1,\tau) - K(t_2,\tau)| K(\tau,s) f(s,u(s),\mu Hg(v,u(v),u''(v))(s))\, ds\, d\tau$$

$$\leq \mu \int_0^1 \int_0^1 |K(t_1,\tau) - K(t_2,\tau)|\, K(\tau,s) ds\, d\tau\, \|u\|_0 \frac{1}{1-L} M_r +$$

$$+ \int_0^1 \int_0^1 |K(t_1,\tau) - K(t_2,\tau)| K(\tau,s)\, f_1(s) f_2(\mu u(s) Hg(v,u(v),u''(v)))\, ds\, d\tau$$

$$\leq \mu 2 |t_1 - t_2| \int_0^1 K(s,s)\, ds\, \|u\|_0 \frac{1}{1-L} M_r +$$

$$+ 2M_d |t_1 - t_2| \int_0^1 \int_0^1 K(s,s)\, f_1(s)\, ds\, d\tau$$

$$\leq 2(\mu D_3 d M + M_d D_4)\, |t_1 - t_2|\, .$$

We have a similar type inequality for $|(Tu)''(t_1) - (Tu)''(t_2)|$.

Therefore $T(V)$ is equicontinuous.

Next, we prove that T is continuous. Suppose $u_n, u \in P \cap (\overline{B}_R / B_r)$ and $\|u_n - u\|_2 \to 0$ which implies that $u_n(t) \to u(t), u_n''(t) \to u''(t)$ uniformly on $[0,1]$. Similarly for $f(t,u,v) \leq f_1(t) f_2(|u| + |v|)$, $f_2(|u_n(t)| + |u_n''(t)|) \to f_2(|u(t)| + |u''(t)|)$ uniformly on $[0,1]$ and $g_1(t,u_n(t)) \to g_1(t,u(t))$ uniformly on $[0,1]$. The assertion follows from the estimate

$$|Tu_n(t) - Tu(t)|$$

$$\leq \mu \int_0^1 \int_0^1 K(t,\tau)K(\tau,s) \mid u_n(s)Hg(v,u_n(v),u_n''(v)) - u(s)Hg(v,u(v),u''(v)) \mid ds\,d\tau +$$

$$+\int_0^1 \int_0^1 K(t,\tau)K(\tau,s)f_1(s) \mid f_2(\mu u_n(s)Hg(v,u_n(v),u_n''(v))) - f_2(\mu u(s)Hg(v,u(v),u''(v))) \mid ds\,d\tau,$$

and the similar estimate for $|(Tu_n)''(t)-(Tu)''(t)|$ by an application of the standard theorem on the convergence of integrals. The Ascoli-Arzela theorem guarantees that $T: P \to P$ is completely continuous.

This finishes the proof. □

Lemma 8. If $u(0)=u(1)=0$ and $u \in C^2[0,1]$, then $\|u\|_0 \leq \|u''\|_0$, and so, $\|u\|_2 = \|u''\|_0$.

Proof. Since $u(0)=u(1)$, there is a $\alpha \in (0,1)$ such that $u'(\alpha)=0$, and so $u'(t)=\int_\alpha^t u''(s)ds$, $t \in [0,1]$. Hence $|u'(t)| \leq \int_\alpha^t |u''(s)|ds \leq \int_0^1 |u''(s)|ds \leq \|u''\|_0$, $t \in [0,1]$. Thus $\|u'\|_0 \leq \|u''\|_0$. Since $u(0)=0$, we have $u(t)=\int_0^t u'(s)ds$, $t \in [0,1]$, and so $|u(t)| \leq \int_0^1 |u'(s)|ds \leq \|u'\|_0$. Thus $\|u\|_0 \leq \|u'\|_0 \leq \|u''\|_0$. Since $\|u\|_2 = \max\{\|u\|_0, \|u''\|_0\}$ and $\|u\|_0 \leq \|u''\|_0$, we obtain that $\|u\|_2 = \|u''\|_0$.

This finishes the proof. □

Corollary 9. Let $r>0$ and let $u \in \partial B_r \cap P$. Then $\|u\|_2 = \|u''\|_0 = r$.

Corollary 10. $\forall u \in X$, $\|u\|_0 \leq \|u''\|_0 \leq \|u^{(4)}\|_0$.

3. Main results

Theorem 1. Let $(H1)$,$(H2)$,$(H3)$ and $(H4)$ hold. Assume that the following condition holds

$(H5)$

$$\limsup_{w \to 0^+} \frac{f_2(w)}{w} \leq c_1,$$

$$\liminf_{\varphi \to \infty} \min_{t \in [\frac{1}{4},\frac{3}{4}]} \inf_{u \in [0,+\infty)} \frac{f(t,u,\varphi)}{\varphi} \geq c_2,$$

and

$$\liminf_{|v| \to \infty} \min_{t \in [\frac{1}{4},\frac{3}{4}]} \inf_{u \in [0,+\infty)} \frac{g(t,u,v)}{|v|} = \infty,$$

where c_1 and c_2 is positive real number.

Then there exists $\mu^* > 0$, such that if $\mu \in (0,\mu^*]$, then problem (3) has at least one positive solution.

Proof.

We divide the rather long proof into three steps.

(I) Firstly, we will prove that the first part of assumptions (i) of Lemma 1 is satisfied.

To do this, by $(H5)$, there exist $0<r<a$ such that

$$f_2(w) \le c_1 w, \quad \forall w \in [0,r]. \tag{33}$$

Let $u \in \partial B_r \cap P$, by Lemma 8, $\|u\|_0 \le \|u''\|_0$ and by Corollary 9, $\|u\|_2 = \|u''\|_0$, then we have $-u''(t) \le \|u''\|_0 = r$ and $u(t) \le \|u\|_0 \le r, \quad \forall t \in [0,1]$. Also since $u \in P$ we have $-u''(t) \ge K(t,t)\,\|u''\|_0\,,\ u(t) \ge K(t,t)\,\|u\|_0\,$, and $u(t) + |u''(t)| \ge rK(t,t),\ t \in [0,1]$. Let $N_r = \widehat{Q}g_1(v, rK(v,v))$ and $0 < \mu \le \min\{\frac{1-L}{N_r}, \frac{1-L}{N_r(D_1+c_1D_2}\}$.

We now show that

$$0 \le \mu u(t) Hg(v,u(v),u''(v)(t)) \le r, \quad \forall t \in [0,1].$$

To see this, since $\mu \le \frac{1-L}{N_r}$ and by (20), we have

$$\mu u(s)\, Hg(v,u(v),u''(v))(s) \le \mu \|u\|_0 \frac{1}{1-L} Qg(v,u(v),u''(v))(s)$$

$$\le \mu r \frac{1}{1-L} Qg_1(v, rK(v,v)) \le \frac{\mu r}{1-L} \widehat{Q}g_1(v, rK(v,v)) = \frac{\mu r N_r}{1-L} \le r.$$

So, using by (33) we have

$$f_2(\mu\ u(s) Hg(v,u(v),u''(v))(s)) \le c_1 \left(\mu u(s) Hg(v,u(v),u''(v))(s)\right).$$

Thus, by Lemma 3, $(H1)$,$(H2)$,$(H3)$ and $(H4)$, we have

$$Tu(t) = \mu \int_0^1 \int_0^1 K(t,\tau)\, K(\tau,s)\, u(s) Hg(v,u(v),u''(v))(s)\, ds\, d\tau +$$

$$+ \int_0^1 \int_0^1 K(t,\tau) K(\tau,s)\, f(s,u(s),\mu Hg(v,u(v),u''(v))(s))\, ds d\tau$$

$$\le \frac{\mu}{1-L} \int_0^1 \int_0^1 K(\tau,\tau)\, K(\tau,s)\ \|u\|_0\, Qg(v,u(v),u''(v))(s)\, ds\, d\tau +$$

$$+ \int_0^1 \int_0^1 K(\tau,\tau) K(\tau,s) f_1(s) f_2(\mu\ u(s) Hg(v,u(v),u''(v))(s)) ds d\tau$$

$$\le \frac{\mu}{1-L} \int_0^1 \int_0^1 K(\tau,\tau)\, K(\tau,s)\ \|u\|_0\, Qg(v,u(v),u''(v))(s)\, ds\, d\tau +$$

$$+ c_1 \mu \int_0^1 \int_0^1 K(\tau,\tau) K(\tau,s) f_1(s) u(s) Hg(v,u(v),u''(v))(s) ds d\tau$$

$$\leq \frac{\mu}{1-L}\int_0^1\int_0^1 K(\tau,\tau)\,K(\tau,s)\,\|u\|_0\,Qg(v,u(v),u''(v))(s)\,ds\,d\tau+$$

$$+c_1\frac{\mu}{1-L}\int_0^1\int_0^1 K(\tau,\tau)K(\tau,s)f_1(s)\,\|u\|_0\,Qg(v,u(v),u''(v))(s)dsd\tau$$

$$\leq \frac{\mu}{1-L}\int_0^1\int_0^1 K(\tau,\tau)K(\tau,s)dsd\tau\,\|u\|_0\,N_r+$$

$$+c_1\frac{\mu}{1-L}\int_0^1\int_0^1 K(\tau,\tau)K(\tau,s)f_1(s)\,\|u\|_0\,N_r dsd\tau$$

$$\leq \frac{\mu N_r(D_1+c_1D_2)}{1-L}\,\|u\|_0 \leq \|u\|_0\,,\quad \forall u\in\partial B_r\cap P,\ \ t\in[0,1].$$

Consequently,

$$\|Tu\|_0 \leq \|u\|_0 \leq \left\|u''\right\|_0, \qquad \forall u\in\partial B_r\cap P. \tag{34}$$

We have a similar type inequality for $\|(Tu)''\|_0$:

$$\|(Tu)''\|_0 \leq \left\|u''\right\|_0, \qquad \forall u\in\partial B_r\cap P. \tag{35}$$

This proves one of assumptions appearing Lemma 1.

(II) Secondly, we will prove that the second part of assumptions (i) of Lemma 1 is satisfied. To do this, by condition $(H4)$ there exists $R_2>0$ such that

$$f(t,u,w)\geq c_2w,\ \ \forall u\in R^+,\ \ w\geq R_2,\ \ t\in\left[\frac{1}{4},\frac{3}{4}\right].$$

Let us choose $c_3\geq \frac{1}{c_2\mu\sigma D_5}$. Then by condition $(H5)$, there exists $R_1>\frac{R_2}{\mu c_3\delta_1K_2D_6}>0$ such that,

$$g(t,u,v)\geq c_3|v|,\ \ \forall u\in R^+,\ \ |v|\geq R_1,\ \ t\in\left[\frac{1}{4},\frac{3}{4}\right].$$

Let $R>\max\{\frac{R_1}{\sigma},a\}$. Let $u\in\partial B_R\cap P$, i.e. $\|u''\|_0=R$. Thus, using by (25) we have

$$\min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]} -u''(t)\geq \sigma\left\|u''\right\|_0=\sigma R>R_1,\ \forall u\in\partial B_R\cap P.$$

It is easy to verify that

$$\mu Hg(v,u(v),u''(v))(s) \geq \mu Qg(v,u(v),u''(v))(s)$$

$$\geq \mu \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} G_1(s,v)G_2(v,z)G_3(z,x)g(x,u(x),u''(x))dxdzdv$$

$$\geq \mu \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} G_1(s,v)G_2(v,z)G_3(z,x)c_3R_1dxdzdv$$

$$\geq \mu c_3R_1\delta_1G_1(s,s) \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} G_1(v,v)G_2(v,z)G_3(z,x)dxdzdv$$

$$\geq \mu c_3R_1\delta_1 \min_{s\in[\frac{1}{4},\frac{3}{4}]} G_1(s,s)D_6 = \mu c_3R_1\delta_1K_2D_6 > R_2, \ \ s \in \left[\frac{1}{4},\frac{3}{4}\right],$$

where $K_2 = \min_{s\in[\frac{1}{4},\frac{3}{4}]} G_1(s,s)$.

Then, by Lemma 3, $(H1)$ and $(H5)$, we have

$$(Tu)(\frac{1}{2}) = \mu \int_0^1 \int_0^1 K(\frac{1}{2},\tau)K(\tau,s)u(s)Hg(v,u(v),u''(v))(s)\,ds\,d\tau +$$

$$+ \int_0^1 \int_0^1 K(\frac{1}{2},\tau)K(\tau,s)f(s,u(s),\mu Hg(v,u(v),u''(v))(s))\,dsd\tau$$

$$\geq \mu \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} K(\frac{1}{2},\tau)K(\tau,s)u(s)Hg(v,u(v),u''(v))(s)\,ds\,d\tau +$$

$$+ \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} K(\frac{1}{2},\tau)K(\tau,s)f(s,u(s),\mu Hg(v,u(v),u''(v))(s))\,dsd\tau$$

$$\geq \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} K(\frac{1}{2},\tau)K(\tau,s)f(s,u(s),\mu Hg(v,u(v),u''(v))(s))\,dsd\tau$$

$$\geq c_2\mu \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} K(\frac{1}{2},\tau)K(\tau,s)Hg(v,u(v),u''(v))(s))\,dsd\tau$$

$$\geq c_2\mu \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} K(\frac{1}{2},\tau)K(\tau,s)Qg(v,u(v),u''(v))(s))\,dsd\tau$$

$$= c_2\mu \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} K(\frac{1}{2},\tau)K(\tau,s) \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} G_1(s,v)G_2(v,z)G_3(z,x)g(x,u(x),u''(x))dxdzdvdsd\tau$$

$$\geq c_2c_3\mu \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} K(\frac{1}{2},\tau)K(\tau,s) \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} G_1(s,v)G_2(v,z)G_3(z,x)|u''(x)|dxdzdvdsd\tau$$

$$\geq c_2c_3\mu\sigma\|u''\|_0 \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} K(\frac{1}{2},\tau)K(\tau,s)G_1(s,v)G_2(v,z)G_3(z,x)dxdzdvdsd\tau$$

$$\geq c_2c_3\mu\sigma D_5\|u''\|_0 \geq \|u''\|_0,$$

so

$$(Tu)(\frac{1}{2}) \geq \|u''\|_0, \qquad \forall u \in \partial B_R \cap P.$$

Consequently,

$$\|u''\|_0 \leq \|Tu\|_0 \leq \left\|(Tu)''\right\|_0, \qquad \forall u \in \partial B_R \cap P. \tag{36}$$

(III) Finally, we will prove that $T : P \cap (\overline{B}_R \backslash B_r) \to P$ is a completely continuous operator. By Lemma 7, the Ascoli-Arzela theorem guarantees that $T : P \cap (\overline{B}_R \backslash B_r) \to P$ is a completely continuous.

Then due to Lemma 1, by (35) and (36) inequality we see that the problem (4) has at least one positive solution.

This finishes the proof. □

4. Conclusionss

This paper investigates the existence of positive solutions for a nonlinear sixth-order differential system using a fixed point theorem of cone expansion and compression type of norm type. The nonlinear terms may be singular with respect to the time and space variables. The problem comes from the deformation analysis of an elastic circular ring segment in the equilibrium state. The results obtained herein generalize and improve some known results including singular and non-singular cases.

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