Asian Journal of Mathematics and Physics
Volume 6, 2022, Article ID 1

Notes on Generalization of Fibonacci, Lucas Matrix Sequences

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In this study, we first introduce sequences in which they generalize Fibonacci and Lucas sequences, called generalized Fibonacci and generalized Lucas and Horadam sequences. After that, by using them, we establish their generalized matrix sequences. We investigate the properties of these sequences such as Binet formulas, some different types of generating functions, sum formulas. We present some important relationships among generalized Fibonacci, generalized Lucas, Horadam matrix sequences.

Keywords: Fibonacci sequence, Lucas sequence, Horadam sequences, matrix sequences.

I. INTRODUCTION AND PRELIMINARIES

Since Fibonacci wrote his book called Liber Abaci at the beginning of the thirteenth century, his intriguing sequence has fascinated mathematicians through the years not only for its inherent mathematical riches but also for its applications in art, nature, architecture. So there are many papers concern about special second-order sequences such as Fibonacci, Lucas, Jacobsthal, Jacobsthal Lucas, Horadam, Pell, Pell Lucas, etc. The elements of the Fibonacci sequences are obtained by adding the two previous terms, beginning with the values $f_0 = 0,\ f_1 = 1$. The ratio of two consecutive elements of the Fibonacci sequence is called Golden Ratio. It is important for almost every research area. The Lucas sequence $\{l_n\}_{n\in\mathbb{N}}$ was generated by changing initial conditions of the Fibonacci sequence are terms of the sequence $\{2,1,3,4,7,...\}$,defined by the recurrence relation $l_n = l_{n-1} + l_{n-2}$ for $n \geq 2$. Because of the importance of special integer sequences, the researchers generalize them by different methods such as changing initial conditions, adding new parameters to recurrence relations. You can see the generalizations of special integer sequences in all of our references. Our first three references are about generalized Fibonacci and Horadam sequences which are some of the oldest researches. In [1,2,4,8], the authors examined the properties of generalized Fibonacci polynomials. In [5] the authors investigated some properties of the Horadam polynomial sequence. In [6] the authors defined generalized $k-$ Horadam sequences and gave some properties by using determinants. In [7], the author gave some combinatorial identities involving generalized Fibonacci- Lucas by defining a special matrix. T. Koshy wrote a book called Fibonacci and Lucas Numbers with Applications in 2001.

In this paper, we give the relationships among generalized Fibonacci, generalized Lucas, and Horadam numbers, and some basic properties of them such as Binet formula generating functions, D'ocagne, Catalan. Then by using these sequences we generate matrix sequences of them and construct some formulas for these sequences.

Definition 1 *Let $n \geq 0$ any integer and p, q are real numbers, and $p^2 + 4q > 0$. The generalized Fibonacci $\{u_n(p,q)\}_{n\in\mathbb{N}}$, gener-*

alized Lucas $\{v_n(p,q)\}_{n\in\mathbb{N}}$ *and Horadam numbers* $\{h_n(p,q)\}_{n\in\mathbb{N}}$ *are defined by the following recurrence relations respectively*

$$u_n(p,q) = pu_{n-1}(p,q) + qu_{n-2}(p,q), \quad (u_0(p,q)=0,\ u_1(p,q)=1) \tag{1.1}$$
$$v_n(p,q) = pv_{n-1}(p,q) + qv_{n-2}(p,q), \quad (v_0(p,q)=2,\ v_1(p,q)=p)$$
$$h_n(p,q) = ph_{n-1}(p,q) + qh_{n-2}(p,q), \quad (h_0(p,q)=a,\ h_1(p,q)=b)$$

The relations among these sequences are

$$\begin{aligned}
v_n &= pu_n + 2qu_{n-1} = u_{n+1} + qu_{n-1},\\
(p^2+4q)u_n &= pv_n + 2qv_{n-1},\\
u_{2n} &= u_n v_n,\\
h_n &= bu_n + qau_{n-1},\\
(p^2+4q)h_n &= (bp+2aq)v_n + q(2b-ap)v_{n-1},\\
u_{n+1}^2 - qu_n^2 &= \frac{pv_{2n+1} - 4(-q)^{n+1}}{p^2+4q}.
\end{aligned}$$

Theorem 2 *Binet Formula enables us to state generalized Fibonacci, generalized Lucas, and Horadam numbers. The Binet Formulas for the generalized Fibonacci, generalized Lucas, Horadam are given by*

$$\begin{aligned}
u_n &= \frac{r_1^n - r_2^n}{r_1 - r_2},\\
v_n &= r_1^n + r_2^n,\\
h_n &= \frac{(b-ar_2)\,r_1^n - (b-ar_1)\,r_2^n}{r_1 - r_2}.
\end{aligned}$$

where $r_1 = \frac{p+\sqrt{p^2+4q}}{2},\ r_2 = \frac{p-\sqrt{p^2+4q}}{2}$ *and*

$$r_1 + r_2 = p, \quad r_1 - r_2 = \sqrt{p^2+4q}, \qquad r_1 . r_2 = -q.$$

Proof. The first part of the proof can be made by substituting $n=0,1$ in the equation $u_n = c_1 r_1^n + c_2 r_2^n$. The other parts of the proof can be made by using the relations between the sequences. For example,

$$\begin{aligned}
v_n = u_{n+1} + qu_{n-1} &= p\frac{r_1^n - r_2^n}{r_1 - r_2} + 2q\frac{r_1^{n-1} - r_2^{n-1}}{r_1 - r_2}\\
&= \frac{\left(p + \frac{2q}{r_1}\right) r_1^n - \left(p + \frac{2q}{r_2}\right) r_2^n}{r_1 - r_2} = \frac{\frac{r_1 p + 2q}{r_1} r_1^n - \frac{r_2 p + 2q}{r_2} r_2^n}{r_1 - r_2}\\
&= \frac{\frac{r_1^2+q}{r_1} r_1^n - \frac{r_2^2+q}{r_2} r_2^n}{r_1 - r_2} = \frac{\left(\frac{r_1^2 - r_1 r_2}{r_1}\right) r_1^n - \left(\frac{r_2^2 - r_1 r_2}{r_2}\right) r_2^n}{r_1 - r_2}\\
&= r_1^n + r_2^n.
\end{aligned}$$

■

Theorem 3 *(Catalan Property)*

$$\begin{aligned}
u_{n-r}u_{n+r} - u_n^2 &= -(-q)^{n-r} u_r^2\\
v_{n-r}v_{n+r} - v_n^2 &= \left(p^2+4q\right)(-q)^{n-r} u_r^2\\
h_{n+r}h_{n-r} - h_n^2 &= -(b-ar_2)(b-ar_1)(-q)^{n-r} u_r^2
\end{aligned}$$

Theorem 4 *(D' ocagne Property)*

$$u_m u_{n+1} - u_n u_{m+1} = (-q)^n u_{m-n}$$

$$v_m v_{n+1} - v_n v_{m+1} = -\left(p^2+4q\right)(-q)^n v_{m-n}$$

$$h_m h_{n+1} - h_n h_{m+1} = (-q)^{n-1}\left(bh_{m-n+1} - ah_{m-n+2}\right)$$

Theorem 5 *(Generating Functions)*

$$\sum_{n=0}^{\infty} u_n t^n = \frac{t}{1-pt-qt^2}$$

$$\sum_{n=0}^{\infty} v_n t^n = \frac{2-tp}{1-pt-qt^2}$$

$$\sum_{n=0}^{\infty} h_n t^n = \frac{a+t(b-ap)}{1-pt-qt^2}$$

II. *GENERALIZED FIBONACCI, GENERALIZED LUCAS AND HORADAM MATRIX SEQUENCES*

In this section, *generalized Fibonacci* $\{U_n(p,q)\}_{n\in\mathbb{N}}$, *generalized* Lucas $\{V_n(p,q)\}_{n\in\mathbb{N}}$, Horadam $\{H_n(p,q)\}_{n\in\mathbb{N}}$matrix sequences are defined by carrying to matrix theory *generalized Fibonacci, generalized Lucas and Horadam sequences.*

Definition 6 *For any integer $n \geq 1$, the generalized Fibonacci matrix sequence is defined by*

$$U_{n+1}(p,q) = pU_n(p,q) + qU_{n-1}(p,q) \tag{2.1}$$

with initial conditions $U_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $U_1 = \begin{pmatrix} p & 1 \\ q & 0 \end{pmatrix}$.

The *generalized* Lucas *matrix sequence is defined by*

$$V_{n+1}(p,q) = pV_n(p,q) + qV_{n-1}(p,q) \tag{2.2}$$

with initial conditions $V_0 = \begin{pmatrix} p & 2 \\ 2q & -p \end{pmatrix}$, $V_1 = \begin{pmatrix} p^2+2q & p \\ qp & 2q \end{pmatrix}$.

The Horadam *matrix sequence is defined by*

$$H_{n+1}(p,q) = pH_n(p,q) + qH_{n-1}(p,q) \tag{2.3}$$

with initial conditions $H_0 = \begin{pmatrix} b & a \\ qa & \frac{b-ap}{q} \end{pmatrix}$, $H_1 = \begin{pmatrix} pb+qa & b \\ qb & qa \end{pmatrix}$.

Theorem 7 *Generalized Fibonacci $\{U_n(p,q)\}_{n\in\mathbb{N}}$, generalized Lucas $\{V_n(p,q)\}_{n\in\mathbb{N}}$, Horadam $\{H_n(p,q)\}_{n\in\mathbb{N}}$, matrix sequences can be demonstrated by using generalized Fibonacci, generalized, Horadam number sequences as the following*

$$U_n = \begin{pmatrix} u_{n+1} & u_n \\ qu_n & qu_{n-1} \end{pmatrix}, \quad V_n = \begin{pmatrix} v_{n+1} & v_n \\ qv_n & qv_{n-1} \end{pmatrix}, \; H_n = \begin{pmatrix} h_{n+1} & h_n \\ qh_n & qh_{n-1} \end{pmatrix}. \tag{2.4}$$

Proof. The proofs are obtained by the mathematical induction method. ■

Theorem 8 *For any integer $m,n \geq 0$, the generalized Fibonacci matrix sequence has the following properties:*

$$U_1^n = U_n \tag{2.5}$$

$$U_{m+n} = U_m U_n \tag{2.6}$$

Proof. The proofs are made by the mathematical induction method. For the proof of (2.6), let $n=0$, then you can easily see that the statement is verified. Let that is true for $n \leq N$. For $n = N+1$

$$\begin{aligned} U_{m+N+1} &= pU_{m+N} + qU_{m+N-1} \\ &= pU_m U_N + qU_m U_{N-1} \\ &= U_m(pU_N + qU_{N-1}) \\ &= U_m U_{N+1}. \end{aligned}$$

■

Corollary 9

$$u_{n+m} = u_m u_{n+1} + qu_{m-1} u_n$$

Proof. By the equality of matrices for the elements of (1,2), it's easily seen. ■

Theorem 10 *For any integer $m,n \geq 0$, the relations between generalized Fibonacci and generalized Lucas sequences are obtained as*

$$U_{n+1} + qU_{n-1} = V_n \tag{2.7}$$

$$(p^2+4q)U_n = pV_n + 2qV_{n-1} \tag{2.8}$$

$$H_n = bU_n + qaU_{n-1} \tag{2.9}$$

$$(p^2+4q)H_n = (bp+2aq)V_n + q(2b-ap)V_{n-1} \tag{2.10}$$

$$V_{n+1} = V_1 U_n \tag{2.11}$$

$$U_m V_{n+1} = V_{n+1} U_m \tag{2.12}$$

Proof. For the proof of (2.7), let $k=1$, the statement is true:

$$U_2 + qU_0 = \begin{bmatrix} p^2+q & p \\ qp & q \end{bmatrix} + \begin{bmatrix} q & 0 \\ 0 & q \end{bmatrix} = \begin{bmatrix} p^2+2q & p \\ qp & 2q \end{bmatrix} = V_1$$

Let us suppose the formula is true until k. We want to show that it is also true for $n = k+1$:

$$\begin{aligned} U_{k+2} + qU_k &= pU_{k+1} + qU_k + q(pU_{k-1} + qU_{k-2}) \\ &= p(U_{k+1} + qU_{k-1}) + q(U_k + qU_{k-2}) \\ &= pV_k + qV_{k-1} = V_{k+1} \end{aligned}$$

For the proof of (2.8), we also use the induction method. For $N = 0$, the truth of the statement is easily seen. For $N = 1$, we get

$$\begin{aligned} V_2 = V_1U_1 &= \begin{pmatrix} p^2+2q & p \\ qp & 2q \end{pmatrix} \cdot \begin{pmatrix} p & 1 \\ q & 0 \end{pmatrix} \\ &= \begin{pmatrix} p^3+3qp & p^2+2q \\ qp^2+2q^2 & qp \end{pmatrix} = \begin{pmatrix} v_3 & v_2 \\ qv_2 & qv_1 \end{pmatrix}. \end{aligned}$$

Let us suppose the formula is true until N. For $n = N+1$,

$$\begin{aligned} V_1U_{N+1} &= V_1U_NU_1 = V_{N+1}U_1 \\ &= \begin{pmatrix} v_{N+2} & v_{N+1} \\ qv_{N+1} & qv_N \end{pmatrix} \cdot \begin{pmatrix} p & 1 \\ q & 0 \end{pmatrix} \\ &= \begin{pmatrix} v_{N+3} & v_{N+2} \\ qv_{N+2} & qv_{N+1} \end{pmatrix}. \end{aligned}$$

For the proof of (2.12), if we consider the initial condition $U_0 = I_2$, then it appears

$$\begin{aligned} U_mV_1U_n &= U_m(pU_1 + 2qU_0)U_n \\ &= (pU_{m+1} + 2qU_m)U_n \\ &= pU_{m+n+1} + 2qU_{m+n} \\ &= pU_nU_1U_m + 2qU_nU_m \\ &= (pU_nU_1 + +2qU_nU_0)U_m \\ &= (pU_1 + 2qU_0)U_nU_m \\ &= V_{n+1}U_m \end{aligned}$$

The other proofs are made by following similar procedures. ■

Theorem 11 *For any integer $n \geq 0$, the following equalities are verified:*

$$H_n = pbU_n + 2aqU_{n-1}, \tag{2.13}$$

$$H_{n+m} = U_nH_m, \tag{2.14}$$

$$H_{n+1}^m = H_1^mU_{mn}. \tag{2.15}$$

Proof. We only show the proof of (2.14). The other proofs are made by following similar procedure. For $n = 0$, the truth of the statement is easily seen. Let that is true for $n \leq m$. For $n = m+1$,

$$H_{n+m+1} = pH_{n+m} + qH_{n+m-1} = pU_nH_m + qU_nH_{m-1} = U_n(pH_m + qH_{m-1}) = U_nH_{m+1}.$$

■

Theorem 12 ***(Binet Formulas)***

$$U_n = \left(\frac{U_1 - r_2 U_0}{r_1 - r_2}\right) r_1^n - \left(\frac{U_1 - r_1 U_0}{r_1 - r_2}\right) r_2^n, \tag{2.16}$$

$$V_{n+1} = \left(\frac{V_2 - r_2 V_1}{r_1 - r_2}\right) r_1^n - \left(\frac{V_2 - r_1 V_1}{r_1 - r_2}\right) r_2^n, \tag{2.17}$$

$$H_{n+1} = \left(\frac{H_2 - r_2 H_1}{r_1 - r_2}\right) r_1^n - \left(\frac{H_2 - r_1 H_1}{r_1 - r_2}\right) r_2^n. \tag{2.18}$$

Proof.

$$\begin{aligned}
&\left(\frac{U_1 - r_2 U_0}{r_1 - r_2}\right) r_1^n - \left(\frac{U_1 - r_1 U_0}{r_1 - r_2}\right) r_2^n \\
&= \frac{r_1^n}{r_1 - r_2}\begin{pmatrix} p - r_2 & 1 \\ q & -r_2 \end{pmatrix} - \frac{r_2^n}{r_1 - r_2}\begin{pmatrix} p - r_1 & 1 \\ q & -r_1 \end{pmatrix} \\
&= \frac{1}{r_1 - r_2}\begin{pmatrix} p\left(r_1^n - r_2^n\right) - r_1 r_2\left(r_1^{n-1} - r_2^{n-1}\right) & r_1^n - r_2^n \\ q\left(r_1^n - r_2^n\right) & -r_1 r_2\left(r_1^{n-1} - r_2^{n-1}\right) \end{pmatrix} \\
&= U_n
\end{aligned}$$

The second and third formulas can be clearly seen by using the identities $V_{n+1} = V_1 U_n,\ H_{n+1} = H_1 U_n$. ■

Theorem 13 ***(Generating Function)***

$$\sum_{n=1}^{\infty} U_n x^n = \frac{U_0 + (U_1 - pU_0)x}{1 - px - qx^2}, \tag{2.19}$$

$$\sum_{n=1}^{\infty} V_n x^n = \frac{V_0 + (V_1 - pV_0)x}{1 - px - qx^2}, \tag{2.20}$$

$$\sum_{n=1}^{\infty} H_n x^n = \left(\frac{H_1 + H_0(r_1 - p)}{r_1 - r_2}\right) r_1^n - \left(\frac{H_1 + H_0(r_2 - p)}{r_1 - r_2}\right) r_2^n. \tag{2.21}$$

Theorem 14 *The determinants of generalized Fibonacci, generalized Lucas, Horadam matrix sequences and by using these determinants, the Simpson formulas for generalized Fibonacci, generalized Lucas, Horadam number sequences are as*

$$\det U_n = (-q)^n,$$
$$u_{n+1}u_{n-1} - u_n^2 = (-1)^n q^{n-1}.$$

$$\det V_{n+1} = q\left(p^2 + 4q\right)(-q)^n,$$
$$v_{n+1}v_{n-1} - v_n^2 = \left(p^2 + 4q\right)(-q)^n.$$

$$\det H_{n+1} = \left(pbqa + q^2a^2 + b^2q\right)(-q)^n,$$
$$h_{n+1}h_{n-1} - h_n^2 = \left(pbq + qa^2 + b^2\right)(-q)^n.$$

Proof. The proofs are made by using the properties of (2.4), (2.5), (2.9), (2.11).

$$\det U_1 = -q \;\text{ and }\; \det U_n = \det U_1^n = (-q)^n = q(u_{n+1}u_{n-1} - u_n^2).$$

$$\det V_{n+1} = \det V_1 \det U_n = q\left(p^2+4q\right)(-q)^n.$$

■

Theorem 15 *For $n \geq 0$, some relationships between the sequences $\{U_n(p,q)\}_{n\in\mathbb{N}}$ and $\{V_n(p,q)\}_{n\in\mathbb{N}}$ are given as*

$$\begin{aligned} V_{n+1}^2 &= V_1^2 U_{2n}, \\ V_{n+1}^2 &= V_1 V_{2n+1}, \\ V_{2n+1} &= U_n V_{n+1}. \end{aligned}$$

Proof. By using the property (2.11),

$$\begin{aligned} V_{n+1}^2 &= V_{n+1}V_{n+1} = V_1U_nV_1U_n = V_1^2U_{2n}, \\ V_{n+1}^2 &= V_1^2U_{2n} = V_1V_1U_{2n} = V_1V_{2n+1}, \\ V_{2n+1} &= V_1U_{2n} = U_nV_{n+1}. \end{aligned}$$

■

Corollary 16

$$\begin{aligned} v_{n+2}^2 + qv_{n+1}^2 &= \left(p^2+4q\right)u_{2n+3}, \\ v_{n+2}^2 + qv_{n+1}^2 &= v_{2n+4} + qv_{2n+2}, \\ v_{2n+1} &= u_{n+1}v_{n+1} + qv_nu_n. \end{aligned}$$

Proof. The proofs are seen by the equality of matrices in Theorem 15. ■

Corollary 17 *For $n \geq 0$,*

$$\begin{aligned} u_{n+1}^2 + \left(1+q^2\right)u_n^2 + q^2u_{n-1}^2 &\geq v_{2n} \\ v_{n+2}^2 + \left(1+q^2\right)v_{n+1}^2 + q^2v_n^2 &\geq v_{2n+4} + 2qv_{2n+2} + q^2v_{2n} \end{aligned}$$

Proof.

$$\left\| \begin{array}{cc} u_{n+1} & u_n \\ qu_n & qu_{n-1} \end{array} \right\|_2^2 \geq \left\| \begin{array}{cc} r_1^n & 0 \\ 0 & r_2^n \end{array} \right\|_2^2$$
$$u_{n+1}^2 + (1+q^2)u_n^2 + q^2u_{n-1}^2 \geq r_1^{2n} + r_2^{2n} = v_{2n}$$

$$V_{n+1} = V_1U_n = (U_2 + qU_0)U_n = U_{n+2} + qU_n.$$

$$\left\| \begin{array}{cc} v_{n+2} & v_{n+1} \\ qv_{n+1} & qv_n \end{array} \right\|_2^2 \geq \left\| \begin{array}{cc} r_1^{n+2} + qr_1^n & 0 \\ 0 & r_2^{n+2} + qr_2^n \end{array} \right\|_2^2$$

$$\begin{aligned}v_{n+2}^2+\left(1+q^2\right)v_{n+1}^2+q^2v_n^2 &\geq \left(r_1^{2n+4}+r_2^{2n+4}\right)+2q\left(r_1^{2n+2}+r_2^{2n+2}\right)+q^2\left(r_1^{2n}+r_2^{2n}\right)\\ &= v_{2n+4}+2qv_{2n+2}+q^2v_{2n}\end{aligned}$$

■

Theorem 18 *For $n\geq 0$,the sum of the first $n+1$ terms of the generalized Fibonacci matrix sequence is obtained by*

$$\sum_{i=0}^{n}U_i=\frac{1}{p+q-1}\begin{bmatrix} u_{n+2}-1+qu_{n+1} & u_{n+2}-1-pu_{n+1}+u_{n+1}\\ qu_{n+1}-q^2u_n-q & qu_{n+1}+pqu_n-qu_n+p-1\end{bmatrix}. \tag{2.22}$$

Theorem 19 *For $m,n,k\geq 0$, and $m\leq n$, the sum formulas for the generalized Fibonacci, the generalized Lucas, Horadam matrix sequences are given as in matrix form*

$$\sum_{i=0}^{k-1}U_{mi+n}=\frac{U_n-(-q)^mU_{n-m}+(-q)^mU_{m(k-1)+n}-U_{mk+n}}{1-v_m+(-q)^m} \tag{2.23}$$

$$\sum_{i=0}^{k-1}V_{mi+n}=\frac{V_n-(-q)^mV_{n-m}+(-q)^mV_{m(k-1)+n}-V_{mk+n}}{1-v_m+(-q)^m} \tag{2.24}$$

$$\sum_{i=0}^{k-1}H_{mi+n}=\frac{H_n-(-q)^mH_{n-m}+(-q)^mH_{m(k-1)+n}-H_{mk+n}}{1-v_m+(-q)^m} \tag{2.25}$$

Proof. Let $A=\frac{U_1-r_2U_0}{r_1-r_2}$, $B=\frac{U_1-r_1U_0}{r_1-r_2}$, then by using the Binet formula we get

$$\begin{aligned}\sum_{i=0}^{k-1}U_{mi+n} &= \sum_{i=0}^{k-1}Ar_1^{mi+n}-Br_2^{mi+n}\\ &= Ar_1^n\left(\frac{1-r_1^{mk}}{1-r_1^m}\right)-Br_2^n\left(\frac{1-r_2^{mk}}{1-r_2^m}\right)\\ &= \frac{Ar_1^n\left(1-r_2^m-r_1^{mk}+(-q)^mr_1^{m(k-1)}\right)-Br_2^n\left(1-r_1^m-r_2^{mk}+(-q)^mr_2^{m(k-1)}\right)}{1-(r_1^m+r_2^m)+(-q)^m}\\ &= \frac{\begin{array}{c}(Ar_1^n-Br_2^n)-(-q)^m\left(Ar_1^{n-m}-Br_2^{n-m}\right)+\\ (-q)^m\left(Ar_1^{m(k-1)+n}-Br_2^{m(k-1)+n}\right)-\left(Ar_1^{mk+n}-Br_2^{mk+n}\right)\end{array}}{1-v_m+(-q)^m}\\ &= \frac{U_n-(-q)^mU_{n-m}+(-q)^mU_{m(k-1)+n}-U_{mk+n}}{1-v_m+(-q)^m}.\end{aligned}$$

■

Theorem 20 *For $m,n,k\geq 0$, and $m\leq n$, $\left|r_1^kr_2^kx\right|<1$ the generating function for the power of the generalized Fibonacci matrix sequence is given as in matrix form*

$$\sum_{i=0}^{\infty}U_i^rx^i=\sum_{i=0}^{\frac{r-1}{2}}\left[(-1)^k\binom{r}{k}\frac{A^{r-k}B^k-A^kB^{r-k}+(-q)^k\left(A^kB^{r-k}r_1^{r-2k}-A^{r-k}B^kr_2^{r-2k}\right)x}{1+(-q)^rx^2-(-q)^k\left(r_2^{r-2k}+r_1^{r-2k}\right)x}\right], for\ r\ is\ odd$$

$$\begin{aligned}\sum_{i=0}^{\infty}U_i^rx^i=&\sum_{i=0}^{\frac{r}{2}-1}\left[(-1)^k\binom{r}{k}(AB)^k\frac{A^{r-2k}-B^{r-2k}+(-q)^k(Br_1)^{r-2k}-(Ar_2)^{r-2k}}{1-(-q)^kV_{r-2k}x+(-q)^rx^2}\right]\\ &+\binom{r}{\frac{r}{2}}\frac{A^{\frac{r}{2}}(-B)^{\frac{r}{2}}}{1-(-q)^{\frac{r}{2}}x} \qquad for\ r\ is\ even\end{aligned}$$

Proof. Let r is odd. Then by using Binet form of U_i and geometric series, we get

$$\begin{aligned}\sum_{i=0}^{\infty}\left(Ar_1^i-Br_2^i\right)^r x^i=\sum_{i=0}^{\infty}U_i^r x^i&=\sum_{i=0}^{\infty}\left[\sum_{k=0}^{r}\binom{r}{k}\left(Ar_1^i\right)^k\left(-Br_2^i\right)^{r-k}\right]x^i\\&=\sum_{k=0}^{r}\binom{r}{k}A^k(-B)^{r-k}\sum_{i=0}^{\infty}\left(r_1^k r_2^{r-k}x\right)^i\\&=\sum_{k=0}^{r}\binom{r}{k}A^k(-B)^{r-k}\frac{1}{1-r_1^k r_2^{r-k}x}\end{aligned}$$

If we divide the sum into two parts contains the equal term in number, we have

$$\begin{aligned}\sum_{i=0}^{\infty}U_i^r x^i&=\sum_{k=0}^{\frac{r-1}{2}}\binom{r}{k}\left(\frac{A^k(-B)^{r-k}}{1-r_1^k r_2^{r-k}x}+\frac{A^{r-k}(-B)^k}{1-r_1^{r-k}r_2^k x}\right)\\&=\sum_{k=0}^{\frac{r-1}{2}}\binom{r}{k}(-1)^k\left(\frac{A^{r-k}B^k}{1-r_1^{r-k}r_2^k x}-\frac{A^k B^{r-k}}{1-r_1^k r_2^{r-k}x}\right)\\&=\sum_{k=0}^{\frac{r-1}{2}}\binom{r}{k}(-1)^k\frac{A^{r-k}B^k-A^kB^{r-k}+\left(A^kB^{r-k}r_1^{r-k}r_2^k-A^{r-k}B^k r_1^k r_2^{r-k}\right)x}{1+(r_1r_2)^r x^2-\left(r_1^k r_2^{r-k}+r_1^{r-k}r_2^k\right)x}\\&=\sum_{i=0}^{\frac{r-1}{2}}\left[(-1)^k\binom{r}{k}\frac{A^{r-k}B^k-A^kB^{r-k}+(-q)^k\left(A^kB^{r-k}r_1^{r-2k}-A^{r-k}B^k r_2^{r-2k}\right)x}{1+(-q)^r x^2-(-q)^k\left(r_2^{r-2k}+r_1^{r-2k}\right)x}\right].\end{aligned}$$

Now, let r is even. If we divide the sum into two parts, we get

$$\begin{aligned}\sum_{i=0}^{\infty}U_i^r x^i&=\sum_{i=0}^{\frac{r}{2}-1}\binom{r}{k}\frac{A^k(-B)^{r-k}}{1-r_1^k r_2^{r-k}x}+\frac{A^{r-k}(-B)^k}{1-r_1^{r-k}r_2^k x}+\binom{r}{\frac{r}{2}}\frac{A^{\frac{r}{2}}(-B)^{\frac{r}{2}}}{1-(-q)^{\frac{r}{2}}x}\\&=\sum_{i=0}^{\frac{r}{2}-1}(-1)^k\binom{r}{k}\frac{A^{r-k}B^k}{1-r_1^{r-k}r_2^k x}+\frac{A^kB^{r-k}}{1-r_1^k r_2^{r-k}x}+\binom{r}{\frac{r}{2}}\frac{A^{\frac{r}{2}}(-B)^{\frac{r}{2}}}{1-(-q)^{\frac{r}{2}}x}\end{aligned}$$

$$\begin{aligned}&=\sum_{i=0}^{\frac{r}{2}-1}(-1)^k\binom{r}{k}\frac{A^{r-k}B^k+A^kB^{r-k}-\left(r_1^k r_2^{r-k}A^{r-k}B^k+A^kB^{r-k}r_1^{r-k}r_2^k\right)x}{1-(-q)^k\left(r_1^{r-2k}+r_2^{r-2k}\right)+(-q)^r x^2}\\&\quad+\binom{r}{\frac{r}{2}}\frac{A^{\frac{r}{2}}(-B)^{\frac{r}{2}}}{1-(-q)^{\frac{r}{2}}x}\\&=\sum_{i=0}^{\frac{r}{2}-1}\left[(-1)^k\binom{r}{k}(AB)^k\frac{A^{r-2k}-B^{r-2k}+(-q)^k(Br_1)^{r-2k}-(Ar_2)^{r-2k}}{1-(-q)^kV_{r-2k}x+(-q)^r x^2}\right]\\&\quad+\binom{r}{\frac{r}{2}}\frac{A^{\frac{r}{2}}(-B)^{\frac{r}{2}}}{1-(-q)^{\frac{r}{2}}x}.\end{aligned}$$

You can find this property for the other matrix sequences by following the same operations. ■

Theorem 21 *For $m,n,k\geq 0$, and $m\leq n$, the generating functions with a negative power for generalized Fibonacci, generalized*

Lucas, Horadam matrix sequences are given as in matrix form

$$\sum_{k=0}^{n}\frac{U_k}{x^k}=\frac{x^2U_0-x\left(-U_1+pU_0\right)}{x^2-px-q}-\frac{U_{n+1}x+qU_n}{x^n\left(x^2-px-q\right)} \tag{2.26}$$

$$\sum_{k=0}^{n}\frac{V_k}{x^k}=\frac{x^2V_1+x\left(V_2-pV_1\right)}{x^2-px-q}-\frac{V_{n+1}x+qV_n}{x^{n-1}\left(x^2-px-q\right)} \tag{2.27}$$

$$\sum_{k=0}^{n}\frac{H_k}{x^k}=\frac{x^2H_1+x\left(H_2-pH_1\right)}{x^2-px-q}-\frac{H_{n+1}x+qH_n}{x^{n-1}\left(x^2-px-q\right)} \tag{2.28}$$

Proof. We give the proof for only the *generalized Fibonacci matrix sequence*

$$\begin{aligned}
\sum_{k=0}^{n}\frac{U_k}{x^k} &= \left(\frac{U_1-r_2U_0}{r_1-r_2}\right)\sum_{k=0}^{n}\left(\frac{r_1}{x}\right)^k-\left(\frac{U_1-r_1U_0}{r_1-r_2}\right)\sum_{k=0}^{n}\left(\frac{r_2}{x}\right)^k \\
&= \frac{1}{x^n}\left(\frac{U_1-r_2U_0}{r_1-r_2}\right)\left(\frac{x^{n+1}-r_1^{n+1}}{x-r_1}\right)-\left(\frac{U_1-r_1U_0}{r_1-r_2}\right)\left(\frac{x^{n+1}-r_2^{n+1}}{x-r_2}\right) \\
&= \frac{1}{x^n\left(x-r_1\right)\left(x-r_2\right)}\left[\frac{\begin{array}{c}\left(U_1-r_2U_0\right)\left(x^{n+1}-r_1^{n+1}\right)\left(x-r_2\right)\\ -\left(U_1-r_1U_0\right)\left(x^{n+1}-r_2^{n+1}\right)\left(x-r_1\right)\end{array}}{r_1-r_2}\right] \\
&= \frac{1}{x^n\left(x^2-px-q\right)}\left[\begin{array}{c}\left(\frac{U_1-r_2U_0}{r_1-r_2}\right)\left(x^{n+2}-r_2x^{n+1}-xr_1^{n+1}+(-q)\,r_1^n\right)\\ -\left(\frac{U_1-r_1U_0}{r_1-r_2}\right)\left(x^{n+2}-r_1x^{n+1}-xr_2^{n+1}+(-q)\,r_2^n\right)\end{array}\right] \\
&= \frac{1}{x^n\left(x^2-px-q\right)}\left[\begin{array}{c}x^{n+2}\left(\frac{U_1-r_2U_0-U_1+r_1U_0}{r_1-r_2}\right)-\\ x^{n+1}\left(\frac{r_2U_1-r_2^2U_0-r_1U_1+r_1^2U_0}{r_1-r_2}\right)\\ -x\left(\left(\frac{U_1-r_2U_0}{r_1-r_2}\right)r_1^{n+1}-\left(\frac{U_1-r_1U_0}{r_1-r_2}\right)r_2^{n+1}\right)\\ -q\left(\left(\frac{U_1-r_2U_0}{r_1-r_2}\right)r_1^{n}-\left(\frac{U_1-r_1U_0}{r_1-r_2}\right)r_2^{n}\right)\end{array}\right] \\
&= \frac{1}{x^n\left(x^2-px-q\right)}\left[U_0x^{n+2}-x^{n+1}\left(-U_1+pU_0\right)-xU_{n+1}-qU_n\right] \\
&= \frac{x^2U_0-x\left(-U_1+pU_0\right)}{x^2-px-q}-\frac{U_{n+1}x+qU_n}{x^n\left(x^2-px-q\right)}
\end{aligned}$$

The other proofs is done by using the properties (2.9) and (2.11). ■

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