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Notes on (s,t)-Pell and (s,t)-Pell Lucas Matrix Sequences

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In this paper, the authors investigated (s,t)-Pell and (s,t)-Pell Lucas matrix sequence in detail and gave some sum formulas and new identities for these sequences. In one of the studies about special integer sequences, Gulec, Taskara proved some identities involving the (s,t)-Pell and (s,t)-Pell Lucas matrix sequences. Their proofs relied heavily on the Binet formula for the (s,t)-Pell and (s,t)-Pell Lucas matrices which was introduced by them. Our goal in this note is to reconsider these identities from another viewpoint and use different proof methods. We also provide other new properties for the (s,t)-Pell and (s,t)-Pell Lucas matrix sequences.
Keywords: *(s,t)*-Pell matrix sequence, *(s,t)*-Pell Lucas matrix sequence, Binet formula.

1. INTRODUCTİON

Special integer sequences are very popular in literature. So you can encounter their different generalizations many times. Both of the most popular of these sequences are Pell and Pell Lucas sequences in literature. You can have detailed knowledge about them in [1-3]. The authors gave some sum formulas and new identities for Pell and Pell Lucas sequences in [4,5,11,15]. Special integer sequences are generalized in many different ways. For example, Civciv and Turkmen carried number sequences to matrix theory and defined the (s, t)-Fibonacci and (s, t)-Lucas matrix sequences in [6,7]. Gulec, Taskara, studied the generalization of other sequences called (s,t)-Pell and (s, t)-Pell-Lucas sequences and they represented their matrix sequences in [8]. Uslu, Uygun, defined the (s,t)-Jacobsthal and (s,t)-Jacobsthal Lucas matrix sequences and their generalizations in [9]. The authors generalized (s,t)-Fibonacci and (s,t)-Lucas sequences and gave properties of the generalized-(s,t) Fibonacci and Fibonacci matrix sequences in [10]. Uygun studied the properties of (s,t)-Jacobsthal and (s,t)-Jacobsthal Lucas sequences in [12] and found some sum formulas of (s,t)-Jacobsthal and (s,t)-Jacobsthal Lucas matrix sequences in [13]. Srisawat and Sriprad denoted (s,t)-Pell and Pell Lucas numbers by using a special matrix in [14]. The authors studied on generalized Fibonacci and k-Pell matrix Sequences in [16]. The authors present many new results for (s,t)-generalized Pell sequence and (s,t)-generalized Pell matrix sequence in [17]. The authors investigated a generalization of the modified Pell sequence, which is called (s,t)-modified Pell sequence. The matrix method is used to get some properties for some sequences of numbers

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The classic Pell sequence is defined as $p_n = 2p_{n-1}+p_{n-2}$ with initial conditions $p_0 = 0$, $p_1 = 1$. The classic Pell Lucas sequence is defined as $q_n = 2q_{n-1}+q_{n-2}$ with initial conditions $q_0 = 2$, $q_1 = 2$.

Gulec, Taskara defined in [8] the (s,t)-Pell sequence $\{p_n(s,t)\}_{n\in\mathbb{N}}$ and (s,t)-Pell-Lucas sequence $\{q_n(s,t)\}_{n\in\mathbb{N}}$ for s, t any real number with $s^2+t>0, n\geq 2$ and $s>0, t\neq 0$ respectively by

$$p_n(s,t)=2sp_{n-1}(s,t)+tp_{n-2}(s,t) \tag{1}$$

$$q_n(s,t)=2sq_{n-1}(s,t)+tq_{n-2}(s,t) \tag{2}$$

with initial conditions $p_0(s,t)=0, p_1(s,t)=1$ and $q_0(s,t)=2, q_1(s,t)=2s$.

The characteristic equation of (1) and (2) is in the form $x^2=2sx+t$ and the roots of the equation are $r_1=s+\sqrt{s^2+t}$ and $r_2=s-\sqrt{s^2+t}$. Note that $r_1+r_2=2s$, $r_1-r_2=2\sqrt{s^2+t}$ and $r_1r_2=-t$. Moreover, it is easily seen that

$$q_n(s,t)=2sp_n(s,t)+2tp_{n-1}(s,t)\ .$$

Their matrix sequences (s,t)-Pell $\{P_n(s,t)\}_{n\in\mathbb{N}}$ and (s,t)-Pell-Lucas $\{Q_n(s,t)\}_{n\in\mathbb{N}}$ are defined in [8] for s, t any real number with $s^2+t>0, n\geq 2$ and $s>0, t\neq 0$ respectively by

$$P_n(s,t)=2sP_{n-1}(s,t)+tP_{n-2}(s,t) \tag{3}$$

$$Q_n(s,t)=2sQ_{n-1}(s,t)+tQ_{n-2}(s,t) \tag{4}$$

with initial conditions

$$P_0(s,t)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, P_1(s,t)=\begin{bmatrix}2s & 1\\ t & 0\end{bmatrix}, Q_0(s,t)=\begin{bmatrix}2s & 2\\ 2t & -2s\end{bmatrix}, Q_1(s,t)=\begin{bmatrix}4s^2+t & 2s\\ 2st & 2t\end{bmatrix}.$$

2. PROPERTIES OF (s,t)-PELL AND (s,t)-PELL LUCAS MATRIX SEQUENCES

Gulec, Taskara proved the following relations for s, t be any real number with $s^2+t>0, n\geq 2, s>0$ and $t\neq 0$:

i. $P_n(s,t)=\begin{bmatrix}p_{n+1}(s,t) & p_n(s,t)\\ tp_n(s,t) & tp_{n-1}(s,t)\end{bmatrix}$, $Q_n(s,t)=\begin{bmatrix}q_{n+1}(s,t) & q_n(s,t)\\ tq_n(s,t) & tq_{n-1}(s,t)\end{bmatrix}$

ii. $P_{m+n}(s,t)=P_n(s,t)P_m(s,t)=P_m(s,t)P_n(s,t)$

iii. $P_n(s,t)=P_1(s,t)^n$

iv. $P_m(s,t)Q_{n+1}(s,t)=Q_{n+1}(s,t)P_m(s,t)=Q_{m+n+1}(s,t)$

v. $Q_{n+1}(s,t)=P_1(s,t)Q_n(s,t)$

vi. $P_n(s,t)=\left(\frac{P_1(s,t)-r_2P_0(s,t)}{r_1-r_2}\right){r_1}^n-\left(\frac{P_1(s,t)-r_1P_0(s,t)}{r_1-r_2}\right){r_2}^n$

vii. $Q_n(s,t)=\left(\frac{Q_1(s,t)-r_2Q_0(s,t)}{r_1-r_2}\right){r_1}^n-\left(\frac{Q_1(s,t)-r_1Q_0(s,t)}{r_1-r_2}\right){r_2}^n$

viii. $Q_{n+1}(s,t)=Q_1(s,t)P_n(s,t)=P_n(s,t)Q_1(s,t)$.

In this paper we give some new properties of (s,t)-Pell and (s,t)-Pell Lucas matrix sequences. By these properties we also derive a number of identities of (s,t)-Pell and (s,t)-Pell Lucas number sequences.

Theorem 1: For $n\geq 0$,we have

$$P_n(s,t)=\begin{bmatrix}\frac{{r_1}^{n+1}-{r_2}^{n+1}}{r_1-r_2} & \frac{{r_1}^n-{r_2}^n}{r_1-r_2}\\ t\frac{{r_1}^n-{r_2}^n}{r_1-r_2} & t\frac{{r_1}^{n-1}-{r_2}^{n-1}}{r_1-r_2}\end{bmatrix} \tag{5}$$

Proof: For $n=1$ the assertion is true $P_1(s,t)=\begin{bmatrix}\frac{{r_1}^{1+1}-{r_2}^{1+1}}{r_1-r_2} & \frac{{r_1}^1-{r_2}^1}{r_1-r_2}\\ t\frac{{r_1}^1-{r_2}^1}{r_1-r_2} & 0\end{bmatrix}=\begin{bmatrix}r_1+r_2 & 1\\ t & 0\end{bmatrix}=\begin{bmatrix}2s & 1\\ t & 0\end{bmatrix}$. For P_1, the eigenvalues are $r_1=s+\sqrt{s^2+t}$ and $r_1=s-\sqrt{s^2+t}$. The eigenvectors of $r_1$and r_2 are $\begin{bmatrix}1\\ -r_2\end{bmatrix}$ and $\begin{bmatrix}1\\ -r_1\end{bmatrix}$. P_1 can be denoted by

$$P_1 = \frac{1}{-r_1+r_2}\begin{bmatrix} 1 & 1 \\ -r_2 & -r_1 \end{bmatrix}\begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}\begin{bmatrix} -r_1 & -1 \\ r_2 & 1 \end{bmatrix}$$
$$= \frac{1}{-r_1+r_2}\begin{bmatrix} r_1 & r_2 \\ -r_1 r_2 & -r_1 r_2 \end{bmatrix}\begin{bmatrix} -r_1 & -1 \\ r_2 & 1 \end{bmatrix}$$
$$= \frac{1}{-r_1+r_2}\begin{bmatrix} -r_1{}^2 + r_2{}^2 & -r_1 + r_2 \\ r_1{}^2 r_2 - r_1 r_2{}^2 & r_1 r_2 - r_1 r_2 \end{bmatrix} = \begin{bmatrix} 2s & 1 \\ t & 0 \end{bmatrix}.$$

Then by using the property $P_n = P_1^n$, it is obtained that

$$P_n = \left(\begin{bmatrix} 1 & 1 \\ -r_2 & -r_1 \end{bmatrix}\begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -r_2 & -r_1 \end{bmatrix}^{-1}\right)^n$$
$$= \frac{1}{-r_1+r_2}.\begin{bmatrix} 1 & 1 \\ -r_2 & -r_1 \end{bmatrix}\begin{bmatrix} r_1{}^n & 0 \\ 0 & r_2{}^n \end{bmatrix}\begin{bmatrix} -r_1 & -1 \\ r_2 & 1 \end{bmatrix}$$
$$= \frac{1}{-r_1+r_2}.\begin{bmatrix} -r_1{}^{n+1} + r_2{}^{n+1} & -r_1{}^n + r_2{}^n \\ r_1 r_2 r_2{}^n - r_1 r_2 r_1{}^n & r_1 r_2 r_2{}^{n-1} - r_1 r_2 r_1{}^{n-1} \end{bmatrix}$$
$$= \frac{1}{r_1-r_2}\begin{bmatrix} r_1{}^{n+1} - r_2{}^{n+1} & r_1{}^n - r_2{}^n \\ t r_1{}^n - t r_2{}^n & t r_1{}^{n-1} - t r_2{}^{n-1} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{r_1{}^{n+1} - r_2{}^{n+1}}{r_1-r_2} & \frac{r_1{}^n - r_2{}^n}{r_1-r_2} \\ t\frac{r_1{}^n - r_2{}^n}{r_1-r_2} & t\frac{r_1{}^{n-1} - r_2{}^{n-1}}{r_1-r_2} \end{bmatrix}.$$

Corollary 2: We obtain the Binet's formula for (s,t)-Pell number sequence as

$$p_n = \frac{r_1{}^n - r_2{}^n}{r_1-r_2}.$$

Corollary 3: We obtain the Binet's formula for (s,t)-Pell Lucas number sequence as

$$q_n = r_1{}^n + r_2{}^n.$$

Proof: The relation between (s,t)-Pell and (s,t)-Pell Lucas sequences is used:

$$q_n = 2sp_n + 2tp_{n-1}$$
$$= 2s\frac{r_1{}^n - r_2{}^n}{r_1-r_2} + 2t\frac{r_1{}^{n-1} - r_2{}^{n-1}}{r_1-r_2}$$
$$= \frac{\left(2s+\frac{2t}{r_1}\right)r_1{}^n - \left(2s+\frac{2t}{r_2}\right)r_2{}^n}{r_1-r_2}$$
$$= \frac{\left(\frac{2sr_1+2t}{r_1}\right)r_1{}^n - \left(\frac{2sr_2+2t}{r_2}\right)r_2{}^n}{r_1-r_2}$$

By the definitions of $r_1,\ r_2$, we have

$$= \frac{\left(\frac{r_1(r_1+r_2)-2r_1r_2}{r_1}\right)r_1{}^n - \left(\frac{r_2(r_1+r_2)-2r_1r_2}{r_2}\right)r_2{}^n}{r_1-r_2}$$
$$= \frac{\left(\frac{r_1{}^2-r_1r_2}{r_1}\right)r_1{}^n - \left(\frac{r_2{}^2-r_1r_2}{r_2}\right)r_2{}^n}{r_1-r_2}$$
$$= \left(\frac{r_1{}^2-r_1r_2}{r_1(r_1-r_2)}\right)r_1{}^n - \left(\frac{r_2{}^2-r_1r_2}{r_2(r_1-r_2)}\right)r_2{}^n$$
$$= \left(\frac{r_1{}^2-r_1r_2}{r_1{}^2-r_1r_2}\right)r_1{}^n - \left(\frac{r_2{}^2-r_1r_2}{r_1r_2-r_2{}^2}\right)r_2{}^n$$
$$= r_1{}^n + r_2{}^n$$

Theorem 4: For $n \in \mathbb{Z}^+$ and $x \in \mathbb{R}$, we have the following formulas for (s,t)-Pell and (s,t)-Pell Lucas sequences

$$\sum_{k=1}^n P_k\, x^{-k} = -\frac{1}{x^n(x^2-2sx-t)}[xP_{n+1} + tP_n] + \frac{1}{(x^2-2sx-t)}[xP_1 + (x^2-2sx)P_0]$$
$$\sum_{k=1}^n Q_k\, x^{-k} = -\frac{1}{x^n(x^2-2sx-t)}[xQ_{n+1} + tQ_n] + \frac{1}{(x^2-2sx-t)}[xQ_1 + (x^2-2sx)Q_0].$$

Corollary 5: For (s,t)-Pell matrix sequence, the following is obtained:

$\sum_{k=1}^{\infty} P_k\, x^{-k} = \frac{1}{(x^2-2sx-t)}\left[xP_1 + (x^2-2sx)P_0\right]$.

Corollary 6: For (s,t)-Pell sequence, the following is derived:

$\sum_{k=1}^{\infty} p_k\, x^{-k} = \frac{x}{(x^2-2sx-t)}$.

Proof: It is known that by Corollary5 and the Property 3-i.

$P_n(s,t) = \begin{bmatrix} p_{n+1}(s,t) & p_n(s,t) \\ tp_n(s,t) & tp_{n-1}(s,t) \end{bmatrix}$.

Using the equality of the matrices, we get

$$\sum_{k=1}^{\infty} P_k\, x^{-k} = \frac{1}{(x^2-2sx-t)}\left[xP_1 + (x^2-2sx)P_0\right]$$

$$= \frac{1}{(x^2-2sx-t)}\left[x\begin{bmatrix} 2s & 1 \\ t & 0 \end{bmatrix} + (x^2-2sx)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]$$

$$= \frac{1}{(x^2-2sx-t)}\begin{bmatrix} 2sx + x^2 - 2sx & x \\ tx & x^2-2sx \end{bmatrix}.$$

Corollary 7: For (s,t)-Pell Lucas matrix sequence, the following is obtained:

$\sum_{k=1}^{\infty} Q_k\, x^{-k} = \frac{1}{(x^2-2sx-t)}\left[xQ_1 + (x^2-2sx)Q_0\right]$.

Corollary 8: For (s,t)-Pell Lucas sequence, the following is derived:

$\sum_{k=1}^{\infty} q_k\, x^{-k} = \frac{2x^2-2sx}{(x^2-2sx-t)}$.

Proof: It is easily seen that by using Corollary 7 and the property 3-i.

Theorem 9: For $\left|r_1^k r_2^{r-k} x\right| < 1$, let be r is any odd positive integer and $A = \left(\frac{P_1 - r_2 P_0}{r_1 - r_2}\right)$, $B = \left(\frac{P_1 - r_1 P_0}{r_1 - r_2}\right)$,

The following sum formula is satisfied:

$\sum_{i=1}^{\infty} P_i^r\, x^i = \sum_{k=0}^{\frac{r-1}{2}}\left[(-1)^k\binom{r}{k}A^k B^k\,\frac{A^{r-2k} - B^{r-2k} + (-t)^k\left(B^{r-2k} r_1^{r-2k} - A^{r-2k} r_2^{r-2k}\right)x}{1-(-t)^k q_{r-2k}x + (-t)^r x^2}\right]$.

Let r is any even positive integer, the following equality derived:

$\sum_{i=1}^{\infty} P_i^r\, x^i = \sum_{k=0}^{\frac{r}{2}-1}\left[(-1)^k\binom{r}{k}A^k B^k\,\frac{A^{r-2k} - B^{r-2k} + (-t)^k\left(B^{r-2k} r_1^{r-2k} - A^{r-2k} r_2^{r-2k}\right)x}{1-(-t)^k q_{r-2k}x + (-t)^r x^2}\right] + \binom{r}{r/2}\frac{A^{r/2}\left(-B^{r/2}\right)}{1-(-t)^{r/2}x}$

Proof: By Binet formula of (s,t)-Pell matrix sequence $P_n = A.r_1{}^n - B.r_2{}^n$, we get

$$\sum_{i=0}^{\infty}\left(Ar_1{}^i - Br_2{}^i\right)^r x^i = \sum_{i=0}^{\infty}\left(\sum_{k=0}^{r}\binom{r}{k}\left(A.r_1{}^i\right)^k\left(-Br_1{}^i\right)^{r-k}\right)x^i$$

$$= \sum_{k=0}^{r}\binom{r}{k}(A)^k(-B)^{r-k}\sum_{i=0}^{\infty}\left(r_1^k r_2^{r-k}x\right)^i$$

$$= \sum_{k=0}^{r}\binom{r}{k}(A)^k(-B)^{r-k}\frac{1}{1-r_1^k r_2^{r-k}x}$$

If r is an odd positive integer, then we have

$$= \sum_{k=0}^{\frac{r-1}{2}}\binom{r}{k}\left(\frac{(A)^k(-B)^{r-k}}{1-r_1^k r_2^{r-k}x} + \frac{(A)^{r-k}(-B)^k}{1-r_1^{r-k}r_2^k x}\right)$$

$$= \sum_{k=0}^{\frac{r-1}{2}}(-1)^k\binom{r}{k}\left(\frac{A^{r-k}B^k}{1-r_1^{r-k}r_2^k x} - \frac{A^k B^{r-k}}{1-r_1^k r_2^{r-k}x}\right)$$

$$= \sum_{k=0}^{\frac{r-1}{2}}(-1)^k\binom{r}{k}\left(\frac{A^{r-k}B^k - A^k B^{r-k} + \left(A^k B^{r-k} r_1^{r-k} r_2^k - A^{r-k}B^k r_1^k r_2^{r-k}\right)x}{1-\left(r_1^k r_2^{r-k} + r_1^{r-k} r_2^k\right)x + (r_1 r_2)^r x^2}\right)$$

$$= \sum_{k=0}^{\frac{r-1}{2}}(-1)^k\binom{r}{k}\left(\frac{A^{r-k}B^k - A^k B^{r-k} + (r_1 r_2)^k\left(A^k B^{r-k} r_1^{r-2k} - A^{r-k}B^k r_2^{r-2k}\right)x}{1-(r_1 r_2)^k\left(r_2^{r-2k} + r_1^{r-2k}\right)x + (r_1 r_2)^r x^2}\right)$$

$$= \sum_{k=0}^{\frac{r-1}{2}}(-1)^k\binom{r}{k}\left(\frac{A^{r-k}B^k-A^kB^{r-k}+(-t)^k\left(A^kB^{r-k}r_1^{r-2k}-A^{r-k}B^kr_2^{r-2k}\right)x}{1-(-t)^k\left(r_2^{r-2k}+r_1^{r-2k}\right)x+(-t)^rx^2}\right)$$

$$= \sum_{k=0}^{\frac{r-1}{2}}(-1)^k\binom{r}{k}A^kB^k\left(\frac{A^{r-2k}-B^{r-k}+(-t)^k\left(B^{r-2k}r_1^{r-2k}-A^{r-2k}r_2^{r-2k}\right)x}{1-(-t)^kq_{r-2k}x+(-t)^rx^2}\right)$$

If r is an even positive integer, then we have,

$$= \sum_{k=0}^{\frac{r}{2}-1}\binom{r}{k}\left(\frac{(A)^k(-B)^{r-k}}{1-r_1^kr_2^{r-k}x}+\frac{(A)^{r-k}(-B)^k}{1-r_1^{r-k}r_2^kx}\right)+\binom{r}{r/2}\frac{A^{r/2}\left(-B^{r/2}\right)}{1-(-t)^{r/2}x}$$

$$= \sum_{k=0}^{\frac{r}{2}-1}(-1)^k\binom{r}{k}\left(\frac{A^{r-k}B^k}{1-r_1^{r-k}r_2^kx}+\frac{A^kB^{r-k}}{1-r_1^kr_2^{r-k}x}\right)+\binom{r}{r/2}\frac{A^{r/2}\left(-B^{r/2}\right)}{1-(-t)^{r/2}x}$$

$$= \sum_{k=0}^{\frac{r}{2}-1}(-1)^k\binom{r}{k}\left(\frac{A^{r-k}B^k+A^kB^{r-k}-\left(A^kB^{r-k}r_1^{r-k}r_2^k+A^{r-k}B^kr_1^kr_2^{r-k}\right)x}{1-\left(r_1^kr_2^{r-k}+r_1^{r-k}r_2^k\right)x+(r_1r_2)^rx^2}\right)+\binom{r}{r/2}\frac{A^{r/2}\left(-B^{r/2}\right)}{1-(-t)^{r/2}x}$$

$$= \sum_{k=0}^{\frac{r}{2}-1}(-1)^k\binom{r}{k}\left(\frac{A^{r-k}B^k+A^kB^{r-k}-(-t)^k\left(A^kB^{r-k}r_1^{r-2k}+A^{r-k}B^kr_2^{r-2k}\right)x}{1-(-t)^k\left(r_2^{r-2k}+r_1^{r-2k}\right)x+t^rx^2}\right)+\binom{r}{r/2}\frac{A^{r/2}\left(-B^{r/2}\right)}{1-(-t)^{r/2}x}$$

$$= \sum_{r=0}^{\frac{r}{2}-1}\left[(-1)^k\binom{r}{k}A^kB^k\frac{A^{r-2k}-B^{r-2k}+(-t)^k\left(B^{r-2k}r_1^{r-2k}-A^{r-2k}r_2^{r-2k}\right)x}{1-(-t)^kq_{r-2k}x+(-t)^rx^2}\right]+\binom{r}{r/2}\frac{A^{r/2}\left(-B^{r/2}\right)}{1-(-t)^{r/2}x}$$

Theorem 10: For $\left|r_1^kr_2^{r-k}x\right|<1$, let be r is any odd positive integer and $X=\left(\frac{Q_1-r_2Q_0}{r_1-r_2}\right)$, $Y=\left(\frac{Q_1-r_1Q_0}{r_1-r_2}\right)$. The following sum formula is satisfied:

$$\sum_{i=1}^{\infty}Q_i^r\,x^i=\sum\sum_{k=0}^{\frac{r-1}{2}}\left[(-1)^k\binom{r}{k}X^kY^k\frac{X^{r-2k}-Y^{r-2k}+(-t)^k\left(Y^{r-2k}r_1^{r-2k}-X^{r-2k}r_2^{r-2k}\right)x}{1-(-t)^kq_{r-2k}x+(-t)^rx^2}\right].$$

Let r is any even positive integer, the following equality obtained:

$$\sum_{i=1}^{\infty}Q_i^r\,x^i=\sum_{k=0}^{\frac{r}{2}-1}\left[(-1)^k\binom{r}{k}X^kY^k\frac{X^{r-2k}-Y^{r-2k}+(-t)^k\left(Y^{r-2k}r_1^{r-2k}-X^{r-2k}r_2^{r-2k}\right)x}{1-(-t)^kq_{r-2k}x+(-t)^rx^2}\right]+\binom{r}{r/2}\frac{X^{r/2}\left(-Y^{r/2}\right)}{1-(-t)^{r/2}x}$$

Proof: The Binet formula of (s,t)-Pell Lucas matrix sequence is $Q_n=Xr_1{}^n-Yr_2{}^n$.

We get the desired result by the same procedure and using Theorem 9 and property vii.

In [8], the authors already gave sum formulas by using Binet formula as

$$\sum_{k=1}^{n}P_{ki+j}=\frac{P_{kn+k+j}+(-t)^kP_{j-k}-(-t)^kP_{kn+j}-P_j}{(x_1)^k+(x_2)^k-(-t)^k-1}$$

$$\sum_{k=1}^{n}Q_{ki+j}=\frac{Q_{kn+k+j}+(-t)^kQ_{j-k}-(-t)^kQ_{kn+j}-Q_j}{(x_1)^k+(x_2)^k-(-t)^k-1}$$

In the following theorem, we present partial sums of (s,t) −Pell matrix and (s,t) −Pell Lucas matrix sequences by using a different proof method.

Theorem 11: The partial sum of (s,t) −Pell matrix sequence for $2s+t\neq 1$ is given in the following

$$\sum_{k=1}^{n}P_k=\frac{1}{2s+t-1}\begin{bmatrix}p_{n+2}+tp_{n+1}-2s-t & p_{n+1}+tp_n-1\\ t(p_{n+1}+tp_n-1) & t(p_n+tp_{n-1}-1)\end{bmatrix}$$

Proof: Let $S_n=\sum_{k=1}^{n}P_k$. By multiplying P_1 both of the sides of the equality and Proposition 3(ii), we get

$S_nP_1=P_2+P_3+P_4+\cdots+P_{n+1.}$

By adding P_1 both of the sides of the equality, it is obtained that

$S_nP_1+P_1=P_1+P_2+P_3+P_4+\cdots+P_{n+1}$

$S_nP_1+P_1=S_n+P_{n+1}$

$S_nP_1-S_n=P_{n+1}-P_1$

$S_n(P_1-P_0)=P_{n+1}-P_1$

The inverse of P_1-P_0 is available for $det(P_1-P_0)=1-2s-t\neq 0$. Then we get

$S_n = (P_{n+1} - P_1)(P_1 - P_0)^{-1}$

By using the following equalities

$P_1 - P_0 = \begin{bmatrix} 2s-1 & 1 \\ t & -1 \end{bmatrix}$, $P_{n+1} - P_1 = \begin{bmatrix} p_{n+2} - 2s & p_{n+1} - 1 \\ tp_{n+1} - t & tp_n \end{bmatrix}$

and

$(P_1 - P_0)^{-1} = \frac{1}{-2s-t+1}\begin{bmatrix} -1 & -1 \\ -t & 2s-1 \end{bmatrix} = \frac{1}{2s+t-1}\begin{bmatrix} 1 & 1 \\ t & 1-2s \end{bmatrix}$,

we get

$S_n = \begin{bmatrix} p_{n+2} - 2s & p_{n+1} - 1 \\ tp_{n+1} - t & tp_n \end{bmatrix}\left(\frac{1}{2s+t-1}\right)\begin{bmatrix} 1 & 1 \\ t & 1-2s \end{bmatrix}$

$= \frac{1}{2s+t-1}\begin{bmatrix} p_{n+2} + tp_{n+1} - 2s - t & p_{n+2} + (1-2s)p_{n+1} - 1 \\ t(p_{n+1} + tp_n - 1) & t(p_{n+1} + (1-2s)p_n - 1) \end{bmatrix}$.

Corollary 12: The partial sum of (s,t) −Pell number sequence for $2s + t \neq 1$ is given in the following

$\sum_{k=1}^{n} p_k = \frac{p_{n+1} + tp_n - 1}{2s+t-1}$.

Proof: It is proved by the equalities of the matrix sequences and from Theorem 11.

Theorem 13: The partial sum of (s,t) −Pell Lucas matrix sequence for $2s + t \neq 1$ is given in the following

$\sum_{k=1}^{n} Q_{k+1} = (a_{ij})$,

$a_{11} = \frac{1}{2s+t-1}[p_{n+4} + tp_{n+3} + 2stp_{n+2} - 4s^2(2s+t) - 2st - t^2 - 1]$

$a_{12} = \frac{1}{2s+t-1}[p_{n+3} + tp_{n+2} - 4s^2 - 2st - t]$

$a_{21} = \frac{1}{2s+t-1}[tp_{n+3} + t^2p_{n+2} + t^2p_{n+1} + t^3p_n - 4s^2t - 2st^2 - 2t^2]$

$a_{22} = \frac{1}{2s+t-1}[tp_{n+2} + t^2p_{n+1} + t^2p_n + t^3p_{n-1} - 4s^2t - 2st - 2t]$

Proof: By using $Q_{k+1} = Q_1P_k$ and Theorem 11, we get

$\sum_{k=1}^{n} Q_{k+1} = \begin{bmatrix} 4s^2 + t & 2s \\ 2st & 2t \end{bmatrix}\frac{1}{2s+t-1}\begin{bmatrix} p_{n+2} + tp_{n+1} - 2s - t & p_{n+1} + tp_n - 1 \\ t(p_{n+1} + tp_n - 1) & t(p_n + tp_{n-1} - 1) \end{bmatrix}$

$a_{11} = \frac{1}{2s+t-1}[4s^2(p_{n+2} + tp_{n+1} - 2s - t) + t(p_{n+2} + tp_{n+1} - 2s - t) + 2st(p_{n+1} + tp_n - 1)]$

$= \frac{1}{2s+t-1}[p_{n+4} + tp_{n+3} + 2stp_{n+2} - 4s^2(2s+t) - 2st - t^2 - 1]$

$a_{12} = \frac{1}{2s+t-1}\left((4s^2 + t)(p_{n+1} + tp_n - 1) + 2st(p_n + tp_{n-1} - 1)\right)$

$= \frac{1}{2s+t-1}[p_{n+3} + tp_{n+2} - 4s^2 - 2st - t]$

$a_{21} = \frac{1}{2s+t-1}[2st(p_{n+2} + tp_{n+1} - 2s - t) + 2t^2(p_{n+1} + tp_n - 1)]$

$= \frac{1}{2s+t-1}[tp_{n+3} + t^2p_{n+2} + t^2p_{n+1} + t^3p_n - 4s^2t - 2st^2 - 2t^2]$

$a_{22} = \frac{1}{2s+t-1}\left(2st(p_{n+1} + tp_n - 1) + 2t^2(p_n + tp_{n-1} - 1)\right)$

$= \frac{1}{2s+t-1}[tp_{n+2} + t^2p_{n+1} + t^2p_n + t^3p_{n-1} - 4s^2t - 2st - 2t]$.

Corollary 14: The partial sum of (s,t) −Pell Lucas number sequence for $2s + t \neq 1$ is given in the following

$\sum_{k=1}^{n} q_{k+1} = \frac{1}{2s+t-1}[p_{n+3} + tp_{n+2} - 4s^2 - 2st - t]$.

Theorem 15: Let $2s + t \neq 1$ and $2s - t \neq 1$, then for $S_{2n} = \sum_{k=1}^{n} P_{2k} = (a_{ij})$ we get

$$a_{11}=\frac{1}{(2s+t-1)(2s-t+1)}(p_{2n+3}-tp_{2n+1}-4s^2+t^2-t),$$

$$a_{12}=\frac{1}{(2s+t-1)(2s-t+1)}(p_{2n+2}-t^2p_{2n}-2s),$$

$$a_{21}=\frac{t}{(2s+t-1)(2s-t+1)}(p_{2n+2}-t^2p_{2n}-2s),$$

$$a_{22}=\frac{t}{(2s+t-1)(2s-t+1)}(p_{2n+1}-t^2p_{2n-1}-1+t).$$

Proof: The proof is made in a similar way in Theorem 11.

Corollary 16: The odd and even elements sums of (s,t) −Pell sequence for $2s+t\neq 1$ and $2s-t\neq 1$ are given in the following:

$$\sum_{k=1}^{n} p_{2k+1}=\frac{1}{(2s+t-1)(2s-t+1)}(p_{2n+3}-tp_{2n+1}-4s^2+t^2-t)$$

$$\sum_{k=1}^{n} p_{2k}=\frac{1}{(2s+t-1)(2s-t+1)}(p_{2n+2}-t^2p_{2n}-2s).$$

Theorem 17: For (s,t) −Pell matrix sequence, the following equality is satisfied:

$$\sum_{i=1}^{n}\binom{n}{i}P_i^r\,x^i=\sum_{k=0}^{r}\binom{r}{k}A^k(-B)^k\left(1+{r_1}^k r_2^{r-k}x\right)^n$$

where $A=\left(\frac{P_1-r_2P_0}{r_1-r_2}\right)$ and $B=\left(\frac{P_1-r_1P_0}{r_1-r_2}\right)$.

Proof: By using the Binet formula of (s,t) −Pell matrix sequence, it is obtained that

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}P_i^r\,x^i&=\sum_{i=0}^{n}\left(A{r_1}^i-B{r_2}^i\right)^r x^i\\&=\sum_{i=0}^{n}\binom{n}{i}\sum_{k=0}^{r}\binom{r}{k}\left(A{r_1}^i\right)^k\left(-B{r_2}^i\right)^{r-k}x^i\\&=\sum_{k=0}^{r}\binom{r}{k}(A)^k(-B)^{r-k}\sum_{k=0}^{r}\binom{n}{i}({r_1}^k{r_2}^{r-k}x)^i\\&=\sum_{k=0}^{r}\binom{r}{k}(A)^k(-B)^{r-k}\,(1+{r_1}^k{r_2}^{r-k}x)^n.\end{aligned}$$

Theorem 18: The partial sum of the product of the consecutive elements of (s,t) −Pell matrix sequence for $i\in\mathbb{Z}$, $t-2s\neq 1$ and $t+2s\neq 1$ is given in the following

$$\sum_{k=0}^{n-1}P_{k+i}\,P_k=\frac{A^2r_1^i(1-r_2^2-r_1^{2n}-r_2^2r_1^{2n})+B^2r_2^i(1-r_1^2-r_2^{2n}-r_1^2r_2^{2n})}{(t-2s-1)(t+2s-1)}-AB\left(\frac{1-(-t)^n}{1+t}\right)q_i$$

where $A=\left(\frac{P_1-r_2P_0}{r_1-r_2}\right)$ and $B=\left(\frac{P_1-r_1P_0}{r_1-r_2}\right)$.

Proof:

$$\sum_{k=0}^{n-1}P_{k+i}\,P_k=\sum_{k=0}^{n-1}\left(Ar_1^{k+i}-Br_2^{k+i}\right)\left(Ar_1^k-Br_2^k\right)$$

$$\begin{aligned}&=\sum_{k=0}^{n-1}A^2r_1^{2k+i}+\sum_{k=0}^{n-1}B^2r_2^{2k+i}-\sum_{k=0}^{n-1}ABr_1^{k+i}r_2^k-\sum_{k=0}^{n-1}ABr_1^k r_2^{k+i}\\&=A^2r_1^i\left(\frac{1-r_1^{2n}}{1-r_1^2}\right)+B^2r_2^i\left(\frac{1-r_2^{2n}}{1-r_2^2}\right)-ABr_1^i\left(\frac{1-(-t)^n}{1+t}\right)-ABr_2^i\left(\frac{1-(-t)^n}{1+t}\right)\\&=\frac{A^2r_1^i(1-r_2^2-r_1^{2n}-r_2^2r_1^{2n})+B^2r_2^i(1-r_1^2-r_2^{2n}-r_1^2r_2^{2n})}{1-(r_1^2+r_2^2)+t^2}-AB\left(\frac{1-(-t)^n}{1+t}\right)(r_1^i+r_2^i)\\&=\frac{A^2r_1^i(1-r_2^2-r_1^{2n}-r_2^2r_1^{2n})+B^2r_2^i(1-r_1^2-r_2^{2n}-r_1^2r_2^{2n})}{(t-2s-1)(t+2s-1)}-AB\left(\frac{1-(-t)^n}{1+t}\right)q_i.\end{aligned}$$

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