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The Truth Values of Experiments Depend on Measurements

Koji Nagata,[1] Do Ngoc Diep,[2] and Tadao Nakamura[3]

[1]*Department of Physics, Korea Advanced Institute of Science and Technology, Daejeon, Korea*

[2]*Institute of Mathematics, Vietnam Academy of Science and Technology, 18 Hoang Quoc Viet road, Cau Giay district, Hanoi, Vietnam*

[3]*Department of Information and Computer Science, Keio University, 3-14-1 Hiyoshi, Kohoku-ku, Yokohama 223-8522, Japan*

We present a different formulation of the Kochen-Specker theorem that relies on the properties of the Kronecker delta notation. This greatly simplifies the Kochen-Specker theorem into a scenario where the Kochen-Specker contradiction can occur with only two measurements. The main result of this paper extends this result into noisy measurements and shows that the contradiction can still be observed. This results to an experimentally accessible version for their simple Kochen-Specker theorem. The results presented here are an interesting and different investigation to the typical Kochen-Specker theorem and we believe they would be of an interest to the community. The report is presented satisfactorily and is rather concise and to the point. We hope gently our discussions could contribute for formulating the Kochen-Specker theorem using the measurement theory based on the truth values into an experimentally accessible theory in terms of a finite precision measurement.

Keywords: Quantum non locality; Quantum measurement theory; Quantum optics; Formalism

I. INTRODUCTION

The great success of quantum mechanics is recognized by the scientific community of physical theories. Einstein, Podolsky, and Rosen discuss the incompleteness argument for quantum mechanics [1]. Many scientists research a hidden-variable interpretation of quantum mechanics. The no-hidden-variable theorem of Bell, Kochen, and Specker is proposed [2, 3]. A strengthened Kochen-Specker theorem, i.e., the free will theorem is discussed by Conway and Kochen [4]. It is begun to research the Kochen-Specker theorem by using inequalities (see Refs. [5–10]). Such inequalities for testing the Kochen-Specker theorem are useful for experimental investigations [11].

Meyer discusses that a finite precision measurement nullifies the Kochen-Specker theorem [12]. Cabello discusses that a finite precision measurement does not nullify the Kochen-Specker theorem [7]. Barrett and Kent give an opinion for the debate [13]. Commutativity, comeasurability, and contextuality in the Kochen-Specker arguments are discussed by Hofer-Szabó [14]. Experimental approach to demonstrating contextuality for qudits is discussed by Sohbi, Ohana, Zaquine, Diamanti, and Markham [15].

The Kochen-Specker theorem based on the Kronecker delta notation is discussed by Nagata, Patro, and Nakamura [16]. Nagata and Nakamura also discuss how quantum mechanics might be when measuring commuting observables based on the property of the Kronecker delta notation [17]. The Kronecker delta notation seems to be necessary for quantum mechanics when using Matrices and Vectors. The Kronecker delta notation is explained as follows: The two-variable function $\delta_{ll'}$ that takes the value 1 when $l = l'$ and the value 0 otherwise. If the elements of a square matrix are defined by the delta function, the matrix produced will be the identity matrix. The name of Kronecker is also used in the Kronecker product. However, in this paper, we dare to use the concept of the Kronecker delta notation.

On the other hand, Nagata and Nakamura propose [18] the measurement theory, in qubits handling, based on the truth values, i.e., the truth T (1) for true and the falsity F (0) for false. The results of measurements are either 0 or 1. They discuss also a classical probability space exists for the measurement theory based on the truth values [19]. We note here this fact is very important for justifying the supposition that the results of measurements 1 and 0 are predetermined "hidden" results of measurements. The Kochen-Specker inequality is violated on the basis of binary logic using the measurement theory based on the truth values [10].

The motivation of the paper is of connecting the measurement theory based on the truth values into an experimentally accessible Kochen-Specker theorem in terms of a finite precision measurement. The number of the necessary results of measurements is only two times. We do not use any Kochen-Specker inequalities. We establish the Kochen-Specker theorem using the measurement theory based on the truth values when the first result is 1 and the second result is 0. Two times are enough. So the proposed theory in this paper is quite simple. How do we perform the experiment of the very simple Kochen-Specker theorem?

Recently, Nagata and Nakamura discuss a novel inconsistency within quantum mechanics when accepting we use the property of the Kronecker delta notation without extra assumptions about the reality of observables [17]. Based on the argumentations, here, we propose an experimentally accessible Kochen-Specker theorem in terms of imperfect sources and detectors.

In more detail, we encounter an imperfect quantum state, the dark count, and the quantum efficiency, which cannot be avoidable from a real experimental situation. If we use the quantum predictions by $2N$ trials, then the Kochen-Specker contradiction increases by an amount that grows linearly with $2N$. In fact, such an error of the number of particles becomes less and less important as we increase trials more and more by using the strong law of large numbers.

We review the definition of the Kochen-Specker contradiction as follows [16], when considering only two measurements. We can depict the predetermined "hidden" results r_1 and r_2 as follows: $r_1 = +x$ and $r_2 = -x$. Let us write V as follows:

$$V = \sum_{l=1}^{2} r_l. \tag{I.1}$$

We evaluate the following value:

$$V \times V = \left(\sum_{l=1}^{2} r_l\right)^2 = \left(\sum_{l=1}^{2} r_l\right) \times \left(\sum_{l'=1}^{2} r_{l'}\right). \tag{I.2}$$

We cannot define $V \times V$ as zero as shown below:

Without the property of the Kronecker delta notation, we have

$$\begin{aligned} V \times V \times \delta_{ll'} &= \left(\sum_{l=1}^{2} r_l\right)^2 \delta_{ll'} \\ &= ((+x) + (-x))^2 \delta_{ll'} = 0 \times \delta_{ll'} = 0, \end{aligned} \tag{I.3}$$

where only the first multiplication is very important. The product does not include the property of the Kronecker delta notation. We derive a necessary condition of the product $(V \times V \times \delta_{ll'})$ of the value V without the property of the Kronecker delta notation. In this case, we have the calculation result as

$$(V \times V \times \delta_{ll'}) = 0. \tag{I.4}$$

This is the necessary condition without the property of the Kronecker delta notation.

In the following, we evaluate the other value of $(V \times V \times \delta_{ll'})$ and derive the other necessary condition within the property of the Kronecker delta notation.

We introduce the property of the Kronecker delta notation then we have

$$
\begin{aligned}
& V \times V \times \delta_{ll'} \\
& = \left(\sum_{l=1}^{2} r_l \right)^2 \times \delta_{ll'} \\
& = \left(\sum_{l=1}^{2} r_l \right) \times \left(\sum_{l'=1}^{2} r_{l'} \right) \times \delta_{ll'} \\
& = (+x)^2 + (-x)^2 = 2x^2,
\end{aligned} \tag{I.5}
$$

where the first and second multiplications are important. The product does include the property of the Kronecker delta notation. Clearly, we have the calculation result as

$$
(V \times V \times \delta_{ll'}) = 2x^2, \tag{I.6}
$$

These argumentations are possible for the case that we utilize the property of the Kronecker delta notation. We cannot assign simultaneously the same two values (“1” and “1”) or (“0” and “0”) for the two suppositions (I.4) and (I.6). Thus, we are in the Kochen-Specker contradiction using two measurements $x, -x$. We symbolize our result as $KS(+x, -x)$

When including observables matters in quantum mechanics analyses, even von Neumann might not mention smoothly the whole analyses that are not always simple in holding consistency of quantum mechanics. There might be representations of some limitation of his analyses as long as he drew his concept even in some too beautiful but difficult way to handle. On the other hand, our trial looks easier to analyze the whole quantum mechanics by virtue of the Kronecker delta notation. This way seems to be easier than his in interpretation. The key point of this paper is to introduce the “Kronecker delta” toward more interpretation-easier analytic power for quantum mechanics itself. Therefore, we are forced to use the same expressions/representations in our writing, in order to make a series of our papers to show our willing opinion.

II. KOCHEN-SPECKER THEOREM USING THE MEASUREMENT THEORY BASED ON THE TRUTH VALUES

We connect the published result of the Kochen-Specker theorem based on the Kronecker delta notation [16] with the measurement theory based on the truth values. We use the quantum mechanics results, that is, ± 1. Here it is worth noting Ref. [17] that the result is again based on the Kronecker delta notation.

We consider the measurement theory, in qubits handling, based on the truth values, i.e., the truth T (1) for true and the falsity F (0) for false. The results of measurements are either 0 or 1 in an ideal case. The first result is 1 and the second result is 0. Surprisingly two results of measurements (1 and 0) are enough to derive our result. We suppose the results of measurements 1 and 0 are predetermined “hidden” results of measurements.

We introduce the map as follows:

$$
g(x) = e^{i\pi x}. \tag{II.1}
$$

The map changes the truth value results into the quantum mechanics results, that is, from $\{0, 1\}$ to $\{1, -1\}$. Then the possible values of the measured results are mapped into either 1 or -1 (quantum mechanics) from either 1 or 0 (the truth values).

Clearly, the quantum mechanics results 1 and -1 must be also predetermined "hidden" results of measurements because the map does not change the classical feature of the truth value results of measurements. However the Kochen-Specker theorem by using the quantum mechanics results 1 and -1 says the quantum mechanics results cannot be predetermined "hidden" results. Therefore, we cannot suppose the truth value results of measurements 1 and 0 are predetermined "hidden" results of measurements. Thus, the Kochen-Specker theorem using the measurement theory based on the truth values is established. We have $KS(+1,-1)$.

Result: *We propose the quite simple Kochen-Specker theorem, where the first result is 1 and then the second result is 0, and vice versa, by virtue of the Kronecker delta notation. The number of the necessary results of measurements is only two times, i.e., 1 and 0.*

We derive the Kochen-Specker theorem using the measurement theory based on the truth values.

III. EXPERIMENTALLY ACCESSIBLE KOCHEN-SPECKER THEOREM USING THE MEASUREMENT THEORY BASED ON THE TRUTH VALUES

In this section, we discuss the main result in this paper. Again, we consider the measurement theory, in qubits handling, based on the truth values, i.e., the truth T (1) for true and the falsity F (0) for false. The results of measurements are either 0 or 1 in an ideal case. The first result is 1 and the second result is 0. Surprisingly two results of measurements (1 and 0) are enough to derive our result. We suppose the results of measurements 1 and 0 are predetermined "hidden" results of measurements.

We introduce the map as follows:

$$f(x) = (-1+\varepsilon)^x - (1-x)\varepsilon. \tag{III.1}$$

The map changes the truth value results into the noisy quantum mechanics results, that is, from $\{0,1\}$ to $\{1-\varepsilon,-1+\varepsilon\}$, where $\varepsilon(<1)$ is interpreted as the reduction factor of the contrast observed in an experiment. Then the possible values of the measured results are mapped into either $1-\varepsilon$ or $-1+\varepsilon$ (quantum mechanics with noise) from either 1 or 0 (the truth values).

Clearly, the noisy quantum mechanics results $1-\varepsilon$ and $-1+\varepsilon$ must be also predetermined "hidden" results of measurements because the map does not change the classical feature of the truth value results of measurements. Then we discuss the Kochen-Specker theorem by using the noisy quantum mechanics results $1-\varepsilon$ and $-1+\varepsilon$. We see this version of the Kochen-Specker theorem is experimentally accessible. It turns out the noisy quantum mechanics results cannot be predetermined "hidden" results. Therefore, we cannot suppose the truth value results of measurements 1 and 0 are predetermined "hidden" results of measurements. Thus, the experimentally accessible Kochen-Specker theorem using the measurement theory based on the truth values is established. We have $KS(+1-\varepsilon,-1+\varepsilon)$.

Result: *We propose the quite simple experimentally accessible Kochen-Specker theorem, where the first result is 1 and then the second result is 0, and vice versa, by virtue of the Kronecker delta notation. The number of the necessary results of measurements is only two times, i.e., 1 and 0.*

We derive the experimentally accessible Kochen-Specker theorem using the measurement theory based on the truth values.

IV. OPTICAL EXPERIMENTS OF THE KOCHEN-SPECKER THEOREM BASED ON THE KRONECKER DELTA NOTATION

We discuss the result of the Kochen-Specker theorem based on the Kronecker delta notation [16] using the noisy quantum mechanics results, i.e., $KS(+1-\epsilon,-1+\epsilon)$ in terms of quantum optics.

Let σ_z be the z-component Pauli observable. It could be defined as follows:

$$\sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{IV.1}$$

Let $|\uparrow\rangle$ and $|\downarrow\rangle$ be eigenstates of σ_z such that $\sigma_z|\uparrow\rangle = +1|\uparrow\rangle$ and $\sigma_z|\downarrow\rangle = -1|\downarrow\rangle$. The measured results of trials are either $+1$ or -1 in the ideal case.

When we consider a quantum optical experiment, we have the following relation with the photon polarization states:

$$\begin{aligned} |\uparrow\rangle &\leftrightarrow |H\rangle, \\ |\downarrow\rangle &\leftrightarrow |V\rangle, \end{aligned} \tag{IV.2}$$

where $|H\rangle$ is a quantum state interpreted by a horizontally polarized photon and $|V\rangle$ is a quantum state interpreted by a vertically polarized photon.

Let us introduce the random noise admixture $\rho_{\text{noise}}(=\frac{1}{2}I)$ into the quantum states, where I is the two-dimensional identity operator. We consider the noisy quantum states emerged from an imperfect source as follows:

$$\begin{aligned} \rho_1 &= (1-\epsilon)|\uparrow\rangle\langle\uparrow| + \epsilon \times \rho_{\text{noise}}, \\ \rho_2 &= (1-\epsilon)|\downarrow\rangle\langle\downarrow| + \epsilon \times \rho_{\text{noise}}. \end{aligned} \tag{IV.3}$$

The value of $\epsilon(<1)$ is interpreted as the reduction factor of the contrast observed in the single-particle experiment. Then we have $\text{tr}[\rho_1\sigma_z] = +1-\epsilon$ and $\text{tr}[\rho_2\sigma_z] = -1+\epsilon$.

Result: *We have been in the Kochen-Specker contradiction when the first result is $+1-\epsilon$ by measuring the Pauli observable σ_z in the quantum state ρ_1, the second result is $-1+\epsilon$ by measuring the same Pauli observable σ_z in the quantum state ρ_2.*

V. DARK COUNT, QUANTUM EFFICIENCY, AND STRONG LAW OF LARGE NUMBERS

In a real experiment, a perfect detector is not feasible. There is an unforeseen effect that an imperfect detector does not count even though the particle indeed passes through the detector (the quantum efficiency). There is also an unforeseen effect that an imperfect detector counts even though the particle does not pass through the detector (the dark count). In this case, we increase measurement outcomes to $2N(\gg 1)$ and then we change such errors into trivial things. If we use the quantum predictions by $2N$ trials, then the Kochen-Specker contradiction increases by an amount that grows linearly with $2N$. In fact, such an error of the number of particles becomes less and less important as we increase trials more and more by using the strong law of large numbers.

We introduce a value V which is the sum of $2N$ data in an experiment. The measured results of trials are either $+1-\epsilon$ or $-1+\epsilon$. We suppose the number of trials of obtaining the result $-1+\epsilon$ is N that is equal to the number (N) of trials of obtaining the result $+1-\epsilon$. We can depict experimental data $r_1, r_2, r_3, \ldots$ as follows: $r_1 = +1-\epsilon$, $r_2 = -1+\epsilon$, $r_3 = +1-\epsilon$ and so on.

Let us write V as follows:

$$V = \sum_{l=1}^{2N} r_l. \tag{V.1}$$

Notice the following value:

$$V \times V = \left(\sum_{l=1}^{2N} r_l \right)^2 = \left(\sum_{l=1}^{2N} r_l \right) \times \left(\sum_{l'=1}^{2N} r_{l'} \right). \tag{V.2}$$

Again, we cannot define $V \times V$ as zero as shown below.

Without using the property of the Kronecker delta notation, we have

$$\begin{aligned} V \times V \times \delta_{ll'} &= \left(\sum_{l=1}^{2N} r_l \right)^2 \delta_{ll'} \\ &= ((+1-\varepsilon) + (-1+\varepsilon) + ... + (-1+\varepsilon))^2 \delta_{ll'} \\ &= 0 \times \delta_{ll'} = 0, \end{aligned} \tag{V.3}$$

where only the first multiplication is very important. The product does not include the property of the Kronecker delta notation. We derive a necessary condition of the product $(V \times V \times \delta_{ll'})$ of the value V. In this case, we have the calculation result as

$$(V \times V \times \delta_{ll'}) = 0. \tag{V.4}$$

This is the necessary condition without using the property of the Kronecker delta notation.

In the following, we derive the other necessary condition for $(V \times V \times \delta_{ll'})$ when we use the property of the Kronecker delta notation. We introduce the property of the Kronecker delta notation then we have

$$\begin{aligned} & V \times V \times \delta_{ll'} \\ &= \left(\sum_{l=1}^{2N} r_l \right)^2 \times \delta_{ll'} \\ &= \left(\sum_{l=1}^{2N} r_l \right) \times \left(\sum_{l'=1}^{2N} r_{l'} \right) \times \delta_{ll'} \\ &= (+1-\varepsilon)^2 + (-1+\varepsilon)^2 + ... + (-1+\varepsilon)^2 \\ &= 2N(+1-\varepsilon)^2, \end{aligned} \tag{V.5}$$

where the first and second multiplications are important. The product does include the property of the Kronecker delta notation. Clearly, we have the calculation result as

$$(V \times V \times \delta_{ll'}) = 2N(+1-\varepsilon)^2. \tag{V.6}$$

We cannot assign simultaneously the same two values (“1” and “1”) or (“0” and “0”) for the two suppositions (V.4) and (V.6). We derive the Kochen-Specker contradiction when we utilize the property of the Kronecker delta notation. If we use the quantum predictions by $2N$ trials, then the Kochen-Specker contradiction increases by an amount that grows linearly with $2N$. In fact, such an error of the number of particles becomes less and less important as we increase trials more and more by using the strong law of large numbers.

Result: *We have been in the Kochen-Specker contradiction when the odd number results are $+1-\varepsilon$ by measuring the Pauli observable σ_z in the quantum state ρ_1, the even number results are $-1+\varepsilon$ by measuring the same Pauli observable σ_z in the quantum state ρ_2.*

VI. CONCLUSIONS

In conclusions, we have proposed an experimentally accessible Kochen-Specker theorem using the measurement theory based on the truth values. We have introduced the measurement theory, in qubits handling, based on the binary logic, i.e., the truth T (1) for true and the falsity F (0) for false.

We have supposed the results of measurements 1 and 0 are predetermined "hidden" results of measurements. Then we have introduced a map that changes the ideal results into the noisy results, that is, $\{0,1\}$ to $\{1-\varepsilon,-1+\varepsilon\}$, where the value of $\varepsilon(<1)$ is interpreted as the reduction factor of the contrast observed in an experiment. Clearly, the noisy results $1-\varepsilon$ and $-1+\varepsilon$ must have been also predetermined "hidden" results of measurements because the map does not change the classical feature of the results of measurements.

Then we have discussed the Kochen-Specker theorem by using the noisy results $1-\varepsilon$ and $-1+\varepsilon$. It has turned out the noisy results cannot be predetermined "hidden" results. Therefore, we cannot have been supposed the results of measurements 1 and 0 are predetermined "hidden" results of measurements. Thus, the Kochen-Specker theorem using the measurement theory based on the truth values has been established. And the theorem has been experimentally feasible because we can use the noisy results.

The number of the necessary results of measurements has been only two times. We do not have used any Kochen-Specker inequalities. We have established the Kochen-Specker theorem using the measurement theory based on the truth values when the first result is 1 and the second result is 0. Two times have been enough. So the proposed theory in this paper has been quite simple. Moreover, we can have performed the experiment of the very simple Kochen-Specker theorem.

We have hoped gently our discussions could contribute for formulating the Kochen-Specker theorem using the measurement theory based on the truth values into an experimentally accessible theory in terms of a finite precision measurement.

It would be interesting to expand upon the possible experimentation aspect of this result. Namely, if the two measurements can be described with POVMs in an explicit manner, and the qubit's state, that together can lead to the expected measurement result in interest.

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DECLARATIONS

Ethical Approval

The authors are in an applicable thought to ethical approval.

Competing Interests

The authors state that there is no conflict of interest.

Author Contributions

Koji Nagata, Do Ngoc Diep, and Tadao Nakamura wrote and read the manuscript.

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