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Dynamical Effects of Anti-predator Behaviour of Adult Prey in a Predator-Prey Model with Ratio-dependent Functional Response

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In this paper, a three species predator-prey model has been developed. Here we divide the prey population into two subpopulations such as (i) juvenile prey and (ii) adult prey population along with only one predator population. It is considered that only adult prey has the anti-predator behaviour. In this paper, two functional responses of predator due to adult and juvenile prey have been introduced on the basis of ratio-dependency. Then the existence condition and boundedness of solution of our proposed mathematical model have been discussed. Also, the different equilibrium points and the stability condition of the system around these equilibrium points have been analyzed. After that, the extinction condition of the prey and predator populations and the effect of anti-predator behaviour on the predator population have been explored. The global stability condition of the proposed system around the positive equilibrium point has been also discussed. Finally, some numerical simulations have been given to test our theoretical results.

Keywords: juvenile prey; adult prey; ratio-dependent functional response; stability analysis; global stability; anti-predator behaviour.

1. INTRODUCTION

The study of interaction between species and their surrounding natural environment is an important topic in theoretical ecology. The systematic mathematical analysis can lead to better understanding of such type of interactions. Since the work of Lotka [4], various kinds of mathematical models about prey-predator interaction [1, 12, 13, 15, 17] have been explored to explain the relationship between prey and predator.

It is natural that two or more species living in a common habitant are often attach to one another by interacting in several ways. In the literatures, the several mathematicians and theoretical ecologists have contributed their different conceptual notions about the growth rate of predator population. Normally, the rate of prey consumption by an average predator is known as functional response which can be classified as (i) prey dependent (ii) predator dependent and (iii) multi species dependent. In prey dependent, the functional response is affected by only prey population, in case of predator dependent, functional response can be determined by considering both predator and prey populations and in multi species dependent the species other than the focal predator and its prey influence the functional response. Traditionally in predator-prey mathematical models, the functional

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response has been considered depending upon density of prey population only. In (1989), Arditi and Ginzburg [14] suggested a ratio dependent functional response which is a particular type of predator dependence. Here, the response only depends on the ratio of prey population size to predator population size. This is better than prey dependent functional response for modelling predation. There are very few number of literatures in mathematical prey-predator models in which the ratio dependent functional responses have been considered. In 2004, the ratio dependent functional response was considered by Fan and Li [18]. After that, in Banerjee [9] developed a prey predator model considering the ratio dependent functional response. In these two papers, a ratio dependency has been considered on Holling type II only. There also some mathematical models [7, 8, 19] in which the ratio-dependent functional response has been considered to analyze those models.

Although biologists routinely label the animals as predators or prey, the ecological role of individuals is often far from clear. There are many examples [2, 6, 16] of role reversals in predators and prey, where an adult prey attacks vulnerable young predators. This implies that a juvenile prey that escapes from predation and become adult and then it can kill juvenile predators. The juvenile prey to adult predators results in behavioral changes later in life: after becoming adult, these prey kill juvenile predators at a faster rate than prey that had not been exposed. Anti-predator adaptations are mechanisms developed through evolution that assist prey organisms in their constant struggle against predators. Throughout the animal kingdom, adaptations have evolved for every stage of this struggle. There are very few mathematical model [5] in which anti-predator behaviours have been considered to analyze the nonlinear system.

In this paper, a three species predator-prey (i.e., juvenile prey, adult prey and predator) model has been developed mathematically where prey population is divided into two subpopulations such as (i) juvenile prey (ii) adult prey population. The anti-predator behaviour property of adult prey population has been introduced in our proposed mathematical model. Here, a Holling type-IV functional response has been used on the basis of ratio-dependency of prey and predator. In this model the existence condition and boundedness of solutions have been discussed. Also the stabilities of the system around the different equilibrium points have been discussed. Here, the extinction condition of the prey and predator population has been derived and then the effect of anti-predator behaviour on the predator population has been also explored. Finally, some numerical simulations have been given to support our theoretical results.

2. MODEL FORMULATION

It is known that in theoretical ecology there are many researches about the dynamical behavior between predator and prey. But, the study of anti-predator behaviour is very important in ecology due to morphological changes and attack of adult prey. Now, according to the model developed by Tang and Xiao [5], it is seen that the growth rate of predator population has been decreased by a anti-predator behavioral term (ηxy) involving the densities of all prey populations. But from the literature survey [2, 6, 16], it is seen that only the adult prey can save itself from the attack of predator due to its morphological changes. So in our proposed model, only the adult prey has been considered to reduce the growth rate of predators. Due to this reason, here the prey population has been divided into two categories such as juvenile prey and adult prey whose densities are $x(t)$ and $y(t)$ at time t. Here, $z(t)$ be the density of predator population at time t. So, the anti-predator behavior should be ηyz where η is the rate of anti-predator behaviour of adult prey to the predator population.

Again, in population dynamics, the functional response is very important to change the density of prey to be attacked by predator per unit time. Now, for the traditional predator-prey model, the functional response depends upon only density of prey population. But according to Berrymen [3], the predator per capita growth rate should decline with its density also. Therefore, to satisfy the above both criteria a functional response should be a function of prey and predator both. In this regard, the following functional responses of predator for consuming juvenile prey and adult prey should be considered as

$$\frac{\beta_1 xz}{z^2+k_1xz+k_2x^2} \text{ and } \frac{\beta_2 yz}{z^2+k_3yz+k_4y^2}$$

respectively. It is also assumed that intrinsic growth rate of juvenile prey (r) is grater than the portion of juvenile prey who becomes adult (β). Hence, considering above realistic criteria, a predator-prey model has been developed in this paper which is as follows:

$$\left.\begin{array}{rcl} \frac{dx}{dt} &=& rx(1-\frac{x}{k})-\beta x-\frac{\beta_1 xz^2}{z^2+k_1xz+k_2x^2} \\ \frac{dy}{dt} &=& \beta x - dy - \frac{\beta_2 yz^2}{z^2+k_3yz+k_4y^2} \\ \frac{dz}{dt} &=& \frac{\mu\beta_1 xz^2}{z^2+k_1xz+k_2x^2}+\frac{\mu_1\beta_2 yz^2}{z^2+k_3yz+k_4y^2}-d_1z-\eta yz \end{array}\right\} \quad (1)$$

with nonnegative initial conditions $x(0)\geq 0, y(0)\geq 0$ and $z(0)\geq 0$.

Here, the parameters involved in the proposed model are described as follows:

- r: intrinsic growth rate of juvenile prey.
- k: environmental carrying capacity.
- β: portion of juvenile prey who becomes adult.
- β_1: attack rate of predator to the juvenile prey.
- β_2: attack rate of predator to the adult prey.
- μ: conservation rate of predator to consume juvenile prey.
- μ_1: conservation rate of predator to consume adult prey.
- η: rate of anti-predator behaviour of adult prey to the predator.
- d: death rate of adult prey.
- d_1: death rate of predator.
- k_1, k_2, k_3, k_4 are the saturation constants for the functional responses.

3. BOUNDEDNESS OF SOLUTIONS

Theorem 1. All solutions of system (1) which originates in R^3_+ are uniformly bounded. **Proof.** Let us define a function of the following form

$$W = \mu x + \mu_1 y + z \quad (2)$$

Taking time derivative of the above equation, it is obtained that

$$\begin{aligned}\frac{dW}{dt} &= \mu\frac{dx}{dt}+\mu_1\frac{dy}{dt}+\frac{dz}{dt} \\ &= \mu rx(1-\frac{x}{k})-\mu\beta x+\mu_1\beta x-\mu_1 dy-d_1z-\eta yz\end{aligned}$$

Now, introducing a positive number η_1 it is obtained that

$$\begin{aligned}&\frac{dW}{dt}+\eta_1 W \\ &= \mu rx(1-\frac{x}{k})-\mu\beta x+\mu_1\beta x+\eta_1\mu x-\mu_1 dy+\eta_1\mu_1 y-d_1z-\eta yz+\eta_1 z \\ &= \mu rx(1-\frac{x}{k})+(\eta_1\mu+\mu_1\beta-\mu\beta)x+(\mu_1\eta_1-\mu_1 d)y+(\eta_1-d_1)z-\eta yz \\ &= \mu rx(1-\frac{x}{k})+(\eta_1\mu+\mu_1\beta-\mu\beta)x+\mu_1(\eta_1-d)y+(\eta_1-d_1)z-\eta yz\end{aligned}$$

If $\eta_1 = max\{\beta(1-\frac{\mu_1}{\mu}), d, d_1\}$, then, it is obtained that

$$\begin{aligned}\frac{dW}{dt}+\eta_1 W &\leq \mu rx(1-\frac{x}{k}) \\ &\leq \frac{dk(r+\eta_1)^2}{4r}\end{aligned}$$

Then solving the above equation, we have

$$0 \leq W(t) \leq e^{-\eta_1 t}C+\frac{l}{\eta_1}$$

where $l = \frac{dk(r+\eta_1)^2}{4r}$.

Now, taking $t\to\infty$, it is obtained that

$$W(t) \leq \frac{l}{\eta_1}$$

Hence all solutions of the system are bounded in the region

$$\Omega = \{(x,y,z)\in R^3 : W(t) = \frac{l}{\eta_1}+\varepsilon, \varepsilon > 0\}$$

4. EXISTENCE AND UNIQUENESS OF SOLUTIONS

From the system (1), we have

$$\begin{aligned}\frac{dx}{dt} &= f_1(t,x,y,z) \\ \frac{dy}{dt} &= f_2(t,x,y,z) \\ \frac{dz}{dt} &= f_3(t,x,y,z)\end{aligned}$$

where

$$f_1(t,x,y,z) = rx(1-\frac{x}{k}) - \beta x - \frac{\beta_1 xz^2}{z^2+k_1xz+k_2x^2}$$
$$f_2(t,x,y,z) = \beta x - dy - \frac{\beta_2 yz^2}{z^2+k_3yz+k_4y^2}$$
$$f_3(t,x,y,z) = \frac{\mu\beta_1 xz^2}{z^2+k_1xz+k_2x^2} + \frac{\mu_1\beta_2 yz^2}{z^2+k_3yz+k_4y^2} - d_1z - \eta yz$$

From the above system of equations, it is seen that all three functions f_1, f_2 and f_3 are continuous in a 4-dimensional rectangular region R defined by

$$|t-t_0| \leq a, |x-c_1| \leq b_1, |x-c_2| \leq b_2, |x-c_3| \leq b_3$$

where (t_0, c_1, c_2, c_3) is a point of real 4-dimensional (t,x,y,z) space and a, b_1, b_2, b_3 are positive constants.

Lemma 1. If $f(x) = rx(1-\frac{x}{k}) - \beta x$ then its maximum value is $\frac{k(r-\beta)^2}{4r}$.

Proof. Now, we have

$$f(x) = rx(1-\frac{x}{k}) - \beta x$$

For extreme value of $f(x)$, $f'(x) = 0$ gives the extreme point $x = \frac{k(r-\beta)}{2r}$. At the point $x = \frac{k(r-\beta)}{2r}$, $f''(x) = -\frac{2r}{k} < 0$. So the function $f(x)$ has a maximum value at $x = \frac{k(r-\beta)}{2r}$ and the maximum value is $\frac{k(r-\beta)^2}{4r}$.

Now, we have

$$f_1(t,x,y,z) = rx(1-\frac{x}{k}) - \beta x - \frac{\beta_1 xz^2}{z^2+k_1xz+k_2x^2}$$
$$i.e., f_1(t,x,y,z) \leq rx(1-\frac{x}{k}) - \beta x,$$
since all parameters and state variables are positive
$$i.e., f_1(t,x,y,z) \leq \frac{k(r-\beta)^2}{4r}, \text{ using } \textbf{Lemma 1.}$$

Lemma 2. If a and b be two real numbers then $|a-b| \leq |a|+|b|$.

Proof. It is known that

$$-|a| \leq a \leq |a| \text{ and } -|b| \leq b \leq |b|.$$

Adding these two inequalities we have

$$-(|a|+|b|) \leq a+b \leq |a|+|b|$$

So, the above equation implies that

$$|a+b| \leq |a|+|b|, \text{ since } -c \leq a \leq c \Longrightarrow |a| \leq c \quad (3)$$

Now, replacing b by $-b$ in equation (3), we have

$$|a-b| \leq |a|+|-b|$$
$$i.e., |a-b| \leq |a|+|b|$$

In the similar way, using **Theorem 1.** and **Lemma 2.** we can prove that

$$|f_2(t,x,y,z)| \leq \beta l_1 + dm_1$$
and
$$|f_3(t,x,y,z)| \leq \frac{\mu\beta_1 l_1 n_1^2}{n_1^2+k_1l_1n_1+k_2l_1^2} + \frac{\mu_1\beta_2 m_1 n_1^2}{n_1^2+k_3m_1n_1+k_4m_1^2} + d_1n_1 + \eta m_1 n_1$$

Since the state variables are bounded, then there exist three positive real numbers l_1, m_1 and n_1 such that

$$|x(t)| \leq l_1, |y(t)| \leq m_1 \text{ and } |z(t)| \leq n_1$$

Therefore, we can write the following

$$|f_i(t,x,y,z)| \leq M, \text{ for } i = 1,2,3 \text{ and } (t,x,y,z) \in R$$

where

$$M = max\{\frac{k(r-\beta)^2}{4r}, \beta l_1 + dm_1, \frac{\mu\beta_1 l_1 n_1^2}{n_1^2+k_1l_1n_1+k_2l_1^2} + \frac{\mu_1\beta_2 m_1 n_1^2}{n_1^2+k_3m_1n_1+k_4m_1^2} + d_1n_1 + \eta m_1 n_1\}$$

Now, any two points such as (t,x_1,y_1,z_1) and (t,x_2,y_2,z_2) be considered in the rectangular region R. For these two points we have

$$f_1(t,x_1,y_1,z_1) - f_1(t,x_2,y_2,z_2)$$
$$= (r-\beta-\beta_1 z_2^2)(x_1-x_2) - \frac{r}{k}(x_1^2-x_2^2) - \beta_1 x_1(z_1^2-z_2^2)$$
$$\leq (r-\beta-\beta_1 n_1^2)(x_1-x_2),$$
provided that $x_1 > x_2, z_1 > z_2, r > \beta + \beta_1 n_1^2$.
$$i.e., |f_1(t,x_1,y_1,z_1) - f_1(t,x_2,y_2,z_2)| \leq K_1|x_1-x_2|,$$
where $K_1 = (r-\beta-\beta_1 n_1^2)$

In the similar way, it can be proved that

$$|f_2(t,x_1,y_1,z_1) - f_2(t,x_2,y_2,z_2)|$$
$$\leq K_2\{|x_1 - x_2| + |y_1 - y_2|\},$$
where $K_2 = max\{\beta, d\}$

and $|f_3(t,x_1,y_1,z_1) - f_3(t,x_2,y_2,z_2)|$
$$\leq K_3\{|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|\},$$
where $K_3 = max\{(\mu\beta_1 n_1^4), (\mu_1\beta_2 n_1^4 - \eta n_1),$
$(\mu\beta_1 k_1 l_1^2 n_1^2 + \mu_1\beta_2 k_3 m_1^2 n_1^2 - d - \eta m_1)\}$

Hence, the three functions $f_1(t,x,y,z), f_2(t,x,y,z)$ and $f_3(t,x,y,z)$ satisfy the lipchitz's condition if following are satisfied.

$r > \beta + \beta_1 n_1^2, \mu_1\beta_2 n_1^3 > \eta$ and $\mu\beta_1 k_1 l_1^2 n_1^2 + \mu_1\beta_2 k_3 m_1^2 n_1^2$
$> d + \eta m_1$

5. POSSIBLE EQUILIBRIUM POINTS AND THEIR STABILITY ANALYSIS

The equilibrium points of the system (1), can be obtained by satisfying the following three equations

$$\frac{dx}{dt} = 0, \frac{dy}{dt} = 0 \text{ and } \frac{dz}{dt} = 0 \tag{4}$$

Solving these three equations we have five equilibrium points which are as follows

$E_0(0,0,0), E_1(k,0,0), E_2(k(1-\frac{\beta}{r}), \frac{\beta k}{d}(1-\frac{\beta}{r}), 0), E_3(x_1^*, 0, z_1^*), E_4(x_2^*, y_2^*, z_2^*)$

where x_1^*, z_1^* with $\beta = 0$ and (x_2^*, y_2^*, z_2^*) satisfying the equation

$$\mu r x_1^*(1 - \frac{x_1^*}{k}) = d_1 z_1^*$$

and

$$\frac{\mu r x_2^{*2}}{k} + x_2^*(\mu\beta - \mu r - \mu_1\beta) + \eta y_2^* z_2^* + d\mu_1 y_2^* + d_1 z_2^* = 0$$

Again, if (x^*, y^*, z^*) be the equilibrium point in general then after combining the equation (4) we have
i.e.,

$$\frac{\mu r x^{*2}}{k} + x^*(\mu\beta - \mu r - \mu_1\beta) + \eta y^* z^* + d\mu_1 y^* + d_1 z^* = 0 \tag{5}$$

Now, this is a quadratic equation in x^* i.e., for each values of y^* and z^* there may exist two values of x^* which are as follows i.e.,

$$x^* = \frac{-(\mu\beta - \mu r - \mu_1\beta) \pm \sqrt{(\mu\beta - \mu r - \mu_1\beta)^2 - 4\frac{\mu r}{k}(\eta y^* z^* + d\mu_1 y^* + d_1 z^*)}}{\frac{2\mu r}{k}} \tag{6}$$

Lemma 3. If $x^* = 0$ then the system (1) goes to the trivial equilibrium point $(0,0,0)$.

Proof. If $x^* = 0$ then from equation (5) we have

$$\eta y^* z^* + d\mu_1 y^* + d_1 z^* = 0.$$

Since all the parameters and state variables are nonnegative, so the above equation will satisfy if $y^* = 0$ and $z^* = 0$. Hence the system goes to the equilibrium point $(0,0,0)$.

Lemma 4. If $\beta = 0$ i.e., $y^* = 0$ and $z^* = 0$ then the system (1) goes to the equilibrium point $(k,0,0)$ and if $\beta = 0$ i.e., $y^* = 0$ and $z^* \neq 0$ then the system goes to the equilibrium point $(x^*, 0, z^*)$.

Proof. If $\beta = 0$ i.e., $y^* = 0$ and $z^* = 0$ then from equation (6) it is obtained that $x^* = k$. Then the system goes the equilibrium point $(k,0,0)$. If $\beta = 0$ i.e., $y^* = 0$, but $z \neq 0$ then from equation (5) we have

$$\frac{\mu r x^{*2}}{k} - \mu r x^* + d_1 z^* = 0$$

then the system goes to an equilibrium point $(x^*, 0, z^*)$ where x^* and z^* will satisfy the above equation.

Lemma 5. For existence of positive equilibrium point of the system (1), $P_1 > 0$, $P_2 > 0$ and $P_3 > 0$ must hold where $P_1 = (\mu\beta - \mu r - \mu_1\beta)^2 - 4\frac{\mu r}{k}\{\eta yz + d\mu_1 y + d_1 z\}$, $P_2 = \mu r + \mu_1\beta - \mu\beta$ and $P_3 = (\eta y + d_1)z + d\mu_1 y$.

Proof. The equation (5) can be written as

$$a x^{*2} + b x^* + c = 0 \tag{7}$$

where $a = \frac{\mu r}{k}, b = (\mu\beta - \mu r - \mu_1\beta), c = \eta y^* z^* + d\mu_1 y^* + d_1 z^*$. Let α_1 and α_2 be any two roots of the equation (7).

Therefore, we have

$$\alpha_1 = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \alpha_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$$
$$i.e., \alpha_1+\alpha_2 = -\frac{b}{a} \text{ and } \alpha_1\alpha_2 = \frac{c}{a}$$

Now, α_1 and α_2 be a positive real roots of equation (7) if

$$b^2-4ac>0, \alpha_1+\alpha_2 = -\frac{b}{a}>0 \text{ and } \alpha_1.\alpha_2 = \frac{c}{a}>0$$

Now, $b^2-4ac>0$ implies that

$$(\mu\beta-\mu r-\mu_1\beta)^2-4\frac{\mu r}{k}\{\eta yz+d\mu_1 y+d_1 z\}>0$$
$$i.e., (\mu\beta-\mu r-\mu_1\beta)^2>4\frac{\mu r}{k}\{\eta yz+d\mu_1 y+d_1 z\} \quad (8)$$

Again, from $\alpha_1+\alpha_2=-\frac{b}{a}>0$, we have

$$-\frac{(\mu\beta-\mu r-\mu_1\beta)}{\frac{\mu r}{k}}>0$$
$$i.e., \{-\mu\beta+\mu r+\mu_1\beta\}>0$$
$$i.e., \mu r+\mu_1\beta>\mu\beta \quad (9)$$

and finally $\alpha_1.\alpha_2=\frac{c}{a}>0$ implies that

$$\eta yz+d\mu_1 y+d_1 z>0$$
$$i.e., (\eta y+d_1)z+d\mu_1 y>0 \quad (10)$$

Therefore, from equation $(8),(9)$ and (10) the proposed system (1) has positive equilibrium point provided that

$$P_1 = (\mu\beta-\mu r-\mu_1\beta)^2-4\frac{\mu r}{k}\{\eta yz+d\mu_1 y+d_1 z\}>0,$$
$$P_2 = \mu r+\mu_1\beta-\mu\beta>0 \text{ and}$$
$$P_3 = (\eta y+d_1)z+d\mu_1 y>0$$

Hence the proof.

Around any equilibrium point, the system will be locally asymptotically stable if all eigenvalues of the linearized variational matrix are negative or have negative real parts.

Now, the jacobian matrix of the system (1) is given by

$$J(x,y,z) = \begin{pmatrix} L_1 & 0 & L_2 \\ \beta & L_3 & L_4 \\ L_5 & L_6 & L_7 \end{pmatrix} \quad (11)$$

where $L_1 = r-\frac{2rx}{k}-\beta-\frac{\beta_1 z^2}{z^2+k_1xz+k_2x^2}+\frac{\beta_1 xz^2(k_1z+2k_2x)}{(z^2+k_1xz+k_2x^2)^2}$, $L_2 = \frac{\beta_1 xz^2(2z+k_1x)}{(z^2+k_1xz+k_2x^2)^2}-\frac{2\beta_1 xz}{z^2+k_1xz+k_2x^2}$, $L_3 = -d-\frac{\beta_2 z^2}{z^2+k_3yz+k_4y^2}+\frac{\beta_2 yz^2(2k_4y+k_3z)}{(z^2+k_3yz+k_4y^2)^2}$, $L_4 = \frac{\beta_2 yz^2(2z+k_3y)}{(z^2+k_3yz+k_4y^2)^2}-\frac{2\beta_2 yz}{z^2+k_3yz+k_4y^2}$, $L_5 = \frac{\mu\beta_1 z^2}{z^2+k_1xz+k_2x^2}-\frac{\mu\beta_1 xz^2(k_1z+2k_2x)}{(z^2+k_1xz+k_2x^2)^2}$, $L_6 = \frac{\mu_1\beta_2 z^2}{z^2+k_3yz+k_4y^2}-\frac{\mu_1\beta_2 yz^2(2k_4y+k_3z)}{(z^2+k_3yz+k_4y^2)^2}-\eta z$, $L_7 = \frac{2\mu\beta_1 xz}{z^2+k_1xz+k_2x^2}-\frac{\mu\beta_1 xz^2(2z+k_1x)}{(z^2+k_1xz+k_2x^2)^2}+\frac{2\mu_1\beta_2 yz}{z^2+k_3yz+k_4y^2}-\frac{\mu_1\beta_2 yz^2(2z+k_3y)}{(z^2+k_3yz+k_4y^2)^2}-d_1-\eta y$.

Stability Analysis at $E_0(0,0,0)$

Since the system is undefined at $(0,0,0)$ and difficult to study the behavior of the system at that point. To overcome such situation, we modify the model (1) as when $(x,y,z)\neq(0,0,0)$ and $\frac{dx}{dt}=\frac{dy}{dt}=\frac{dz}{dt}=0$ at $E_0(0,0,0)$. To analyze the behavior of the system at trivial equilibrium, we follow the method developed by Arino et al. [11]. Then we rewrite the model as

$$\frac{dV}{dt} = H(V(t))+Q(V(t))$$

where $H(.)$ is a continuous and homogeneous function of degree one; $V=(x,y,z)$; Q is a C^1 function with $Q(V)=o(V)$. For the present problem, $H=((r-\beta)x,-dy,-d_1z)$. Let $V(t)$ be a solution of the above such that $liminf_{t\to\infty}||V(t)||=0$ and $V(t_n)$ be the corresponding sequence which tends to zero as $t\to\infty$.

Define $y_n=(V(t_n+s)/||V(t_n+s)||)$. Then, y_n is a sequence such that $||y_n||=1$. Now, by Ascoli-Arzela theorem, there should exist a subsequence of y_n that converges to a function $y(t)$ satisfying the equation

$$\frac{dy}{dt} = H(y(t))-(y(t),H(y(t)))y(t) \quad (12)$$

The steady state of the above equation will be given by the vector $v(t)=(v_1,v_2,v_3)$ where $H(v)=(v,H)v$ are the solutions of the eigenvalue problem

$$H(v) = \lambda v$$
$$\lambda = (v,H(v)) \quad (13)$$

From the above equation we have $(r-\beta-\lambda)v_1 = 0, (d+\lambda)v_2=0, (d_1+\lambda)v_3=0$.

We now study the following cases:

Case I: $v_1\neq 0, v_2=v_3=0$.

In this case, the system can reach the trivial equilibrium (origin) along the x-axis with $\lambda=r-\beta$ when $r<\beta$.

Case II: $v_1=v_3=0, v_2\neq 0$.

The system will reach the origin along the y-axis with $\lambda = -d$.

Case III: $v_1 = v_2 = 0, v_3 \neq 0$.

The system will reach the origin along the z-axis with $\lambda = -d_1$.

Stability Analysis at $E_1 = (k, 0, 0)$ for $\beta = 0$

Since the system is undefined at $(k, 0, 0)$ and difficult to study the behavior of the system at that point. Stability of the system (1) at the point $(k, 0, 0)$ is analyzed using the same approach used in stability analysis at $(0, 0, 0)$. We have $(-r-\lambda)v_1 = 0, (d+\lambda)v_2 = 0, (d_1+\lambda)v_3 = 0$.

Now, the following cases can be obtained

Case I: $v_1 \neq 0, v_2 = v_3 = 0$.

In this case, the system will reach the equilibrium $(k, 0, 0)$ along the x-axis with $\lambda = -r$.

Case II: $v_1 = v_3 = 0, v_2 \neq 0$.

The system will reach $(k, 0, 0)$ along the y-axis with $\lambda = -d$.

Case III: $v_1 = v_2 = 0, v_3 \neq 0$.

The system will reach $(k, 0, 0)$ along the z-axis with $\lambda = -d_1$.

Lemma 6. There exists an asymptotical stability around the equilibrium point $E_2(k(1-\frac{\beta}{r}), \frac{\beta k}{d}(1-\frac{\beta}{r}), 0)$.

Proof. The jacobian matrix of the system (1) at $E_2(k(1-\frac{\beta}{r}), \frac{\beta k}{d}(1-\frac{\beta}{r}), 0)$ is given by

$$\begin{pmatrix} -r+\beta & 0 & 0 \\ \beta & -d & 0 \\ 0 & 0 & -d_1 - \frac{\eta\beta k}{d}(1-\frac{\beta}{r}) \end{pmatrix}$$

The eigenvalues of the jacobian matrix are $-r+\beta, -d, -d_1 - \frac{\eta\beta k}{d}(1-\frac{\beta}{r})$.

Since $r > \beta$ then all the eigenvalues of the jacobian matrix are negative and the system (1) will be locally asymptotically stable at the equilibrium point E_2.

The following asymptotically stabilities have been discussed about the equilibrium point $E_3(x_1^*, 0, z_1^*)$ for $\beta = 0$ and $E_4(x_2^*, y_2^*, z_2^*)$.

Lemma 7. The system (1) will be an locally asymptotically stable around $E_3(x_1^*, 0, z_1^*)$ for $\beta = 0$ if $c_1 > 0, c_3 > 0$ and $c_1c_2 - c_3 > 0$.

Proof. At the equilibrium point $E_3(x_1^*, 0, z_1^*)$ where $\beta = 0$, corresponding jacobian matrix is given by

$$\begin{pmatrix} M_1 & M_2 & M_3 \\ M_4 & M_5 & M_6 \\ M_7 & M_8 & M_9 \end{pmatrix}$$

where $M_1 = r - \frac{2rx_1^*}{k} - \frac{\beta_1 {z_1}^{*2}}{({z_1}^{*2}+k_1x_1^*z_1^*+k_2{x_1}^{*2})} + \frac{\beta_1 x_1^* {z_1}^{*2}}{({z_1}^{*2}+k_1x_1^*z_1^*+k_2{x_1}^{*2})}, M_2 = 0, M_3 = \frac{\beta_1 x_1^* {z_1}^{*2}(2z_1^*+k_1x_1^*)}{({z_1}^{*2}+k_1x_1^*z_1^*+k_2{x_1}^{*2})^2} - \frac{2\beta_1 x_1^* z_1^*}{({z_1}^{*2}+k_1x_1^*z_1^*+k_2{x_1}^{*2})}, M_4 = 0, M_5 = -d-\beta_2, M_6 = 0.0,\ M_7 = \frac{\mu\beta_1 {z_1}^{*2}}{({z_1}^{*2}+k_1x_1^*z_1^*+k_2{x_1}^{*2})} - \frac{\mu\beta_1 x_1^* {z_1}^{*2}(k_1z_1^*+2k_2x_1^*)}{({z_1}^{*2}+k_1x_1^*z_1^*+k_2{x_1}^{*2})^2},\ M_8 = \mu_1\beta_2 - \eta z_1^*$, $M_9 = \frac{2\mu\beta_1 x_1^* z_1^*}{({z_1}^{*2}+k_1x_1^*z_1^*+k_2{x_1}^{*2})} - \frac{\mu\beta_1 x_1^* {z_1}^{*2}(2z_1^*+k_1x_1^*)}{({z_1}^{*2}+k_1x_1^*z_1^*+k_2{x_1}^{*2})^2} - d_1$.

Then the characteristic equation of the above jacobian matrix is

$$\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 = 0$$

where $c_1 = -(M_1 + M_5 + M_9), c_2 = (M_1M_5 + M_1M_9 + M_5M_9 - M_6M_8 - M_2M_4 - M_3M_7)$ and $c_3 = M_3M_5M_7 + M_2M_4M_9 + M_1M_6M_8 - M_1M_5M_9 - M_2M_6M_7 - M_3M_4M_8$. Now, by Routh-Hurwitch criteria the system (1) will be locally asymptotically stable around E_3 if $c_1 > 0, c_3 > 0$ and $c_1c_2 - c_3 > 0$.

Lemma 8. The system (1) will be locally asymptotically stable around $E_4(x_2^*, y_2^*, z_2^*)$ if $b_1 > 0, b_3 > 0$ and $b_1b_2 - b_3 > 0$.

Proof. At the equilibrium point $E_4(x_2^*, y_2^*, z_2^*)$, the corresponding jacobian matrix is given by

$$\begin{pmatrix} N_1 & N_2 & N_3 \\ N_4 & N_5 & N_6 \\ N_7 & N_8 & N_9 \end{pmatrix}$$

where $N_1 = r - \frac{2rx_2^*}{k} - \beta - \frac{\beta_1 {z_2}^{*2}}{({z_2}^{*2}+k_1x_2^*z_2^*+k_2{x_2}^{*2})} + \frac{\beta_1 x_2^* {z_2}^{*2}}{({z_2}^{*2}+k_1x_2^*z_2^*+k_2{x_2}^{*2})^2}, N_2 = 0, N_3 = \frac{\beta_1 x_2^* {z_2}^{*2}(2z_2^*+k_1x_2^*)}{({z_2}^{*2}+k_1x_2^*z_2^*+k_2{x_2}^{*2})^2} - \frac{2\beta_1 x_2^* z_2^*}{({z_2}^{*2}+k_1x_2^*z_2^*+k_2{x_2}^{*2})}, N_4 = \beta, N_5 = -d - \frac{\beta_2 {z_2}^{*2}}{({z_2}^{*2}+k_3y_2^*z_2^*+k_4{y_2}^{*2})} + \frac{\beta_2 y_2^* {z_2}^{*2}(k_3z_2^*+2k_4y_2^*)}{({z_2}^{*2}+k_3y_2^*z_2^*+k_4{y_2}^{*2})^2}, N_6 = \frac{\beta_2 y_2^* {z_2}^{*2}(k_3y_2^*+2z_2^*)}{({z_2}^{*2}+k_3y_2^*z_2^*+k_4{y_2}^{*2})^2} -$

$\frac{2\beta_2 y_2^* z_2^*}{(z_2{}^{*2}+k_3 y_2^* z_2^*+k_4 y_2{}^{*2})}$, $N_7 = \frac{\mu\beta_1 z_2{}^{*2}}{(z_2{}^{*2}+k_1 x_2^* z_2^*+k_2 x_2{}^{*2})} - \frac{\mu\beta_1 x_2^* z_2{}^{*2}(k_1 z_2^*+2k_2 x_2^*)}{(z_2{}^{*2}+k_1 x_2^* z_2^*+k_2 x_2{}^{*2})^2}$, $N_8 = \frac{\mu_1\beta_2 z_2{}^{*2}}{(z_2{}^{*2}+k_3 y_2^* z_2^*+k_4 y_2{}^{*2})} - \frac{\mu_1\beta_2 y_2^* z_2{}^{*2}(k_3 z_2^*+2k_4 y_2^*)}{(z_2{}^{*2}+k_3 y_2^* z_2^*+k_4 y_2{}^{*2})^2} - \eta z_2^*$, $N_9 = \frac{2\mu\beta_1 x_2^* z_2^*}{(z_2{}^{*2}+k_1 x_2^* z_2^*+k_2 x_2{}^{*2})} - \frac{\mu\beta_1 x_2^* z_2{}^{*2}(2z_2^*+k_1 x_2^*)}{(z_2{}^{*2}+k_1 x_2^* z_2^*+k_2 x_2{}^{*2})^2} + \frac{2\mu_1\beta_2 y_2^* z_2^*}{(z_2{}^{*2}+k_3 y_2^* z_2^*+k_4 y_2{}^{*2})} - \frac{\mu_1\beta_2 y_2^* z_2{}^{*2}(2z_2^*+k_3 y_2^*)}{(z_2{}^{*2}+k_3 y_2^* z_2^*+k_4 y_2{}^{*2})^2} - d_1 - \eta y_2^*$.

Then the characteristic equation of the above jacobian matrix is

$$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0$$

where $b_1 = -(N_1+N_5+N_9), b_2 = (N_1N_5+N_1N_9+N_5N_9-N_6N_8-N_2N_4-N_3N_7)$ and $b_3 = N_3N_5N_7+N_2N_4N_9+N_1N_6N_8-N_1N_5N_9-N_2N_6N_7-N_3N_4N_8$. Now, by Routh-Hurwitch criteria the system (1) will be locally asymptotically stable around E_4 if $b_1 > 0, b_3 > 0$ and $b_1b_2 - b_3 > 0$.

Theorem 2. The three populations i.e., juvenile prey, adult prey and predator populations will extinct provided that $\beta_1 > (r-\beta)$ and $k_1 = k_2 = k_3 = k_4 = 0$.

Proof. We have from the first equation of system (1) with $k_1 = k_2 = 0$, it is obtained that

$$\begin{aligned} &\frac{dx}{dt} = (r-\beta)x - \frac{rx^2}{k} - \beta_1 x \\ &i.e., \frac{dx}{dt} \le (r-\beta)x - \beta_1 x \\ &i.e., \frac{dx}{x} \le (r-\beta-\beta_1)dt \end{aligned} \tag{14}$$

Then integrating equation (14), it is obtained that

$$x \le C_1 e^{(r-\beta-\beta_1)t}$$

Now, if $\beta_1 > (r-\beta)$, then taking t tends to infinity we have $x \to 0$. Again, from the second equation of system (1) with $k_3 = k_4 = 0$, it is obtained that

$$\begin{aligned} &\frac{dy}{dt} = \beta x - dy - \beta_2 y \\ &i.e., \frac{dy}{dt} \le C_1\beta e^{(r-\beta-\beta_1)t} - (d+\beta_2)y \\ &i.e., \frac{dy}{dt} + (d+\beta_2)y \le C_1\beta e^{(r-\beta-\beta_1)t} \end{aligned} \tag{15}$$

Then solving equation (15), it is obtained that

$$y \le \frac{C_1\beta e^{(r-\beta-\beta_1)t}}{(r-\beta-\beta_1+d+\beta_2)} + C_2 e^{-(d+\beta_2)t}$$

Now, if $\beta_1 > (r-\beta)$, then taking t tends to infinity we have $y \to 0$.

Also again, from the third equation of system (1) with $k_1 = k_2 = k_3 = k_4 = 0$, it is obtained that

$$\begin{aligned} &\frac{dz}{dt} = \mu\beta_1 x + \mu_1\beta_2 y - d_1 z - \eta yz \\ &i.e., \frac{dz}{dt} \le C_1\mu\beta_1 e^{(r-\beta-\beta_1)t} + C_1\mu_1\beta_2 \frac{\beta e^{(r-\beta-\beta_1)t}}{(r-\beta-\beta_1+d+\beta_2)} \\ &+C_2\mu_1\beta_2 e^{-(d+\beta_2)t} - d_1 z \\ &i.e., \frac{dz}{dt} + d_1 z \le C_1 e^{(r-\beta-\beta_1)t}\left[\mu\beta_1 + \frac{\mu_1\beta_2}{(r-\beta-\beta_1+d+\beta_2)}\right] \\ &+C_2\mu_1\beta_2 e^{-(d+\beta_2)t} \end{aligned} \tag{16}$$

Then, solving equation (16), it is obtained that

$$z \le C_1 \frac{\left[\mu\beta_1 + \frac{\mu_1\beta_2}{(r-\beta-\beta_1+d+\beta_2)}\right]}{(r-\beta-\beta_1+d_1)} e^{(r-\beta-\beta_1)t} + C_2 \frac{\mu_1\beta_2}{(d+\beta_2+d_1)} e^{-(d+\beta_2)t}$$

Now, if $\beta_1 > (r-\beta)$, then taking t tends to infinity we have $z \to 0$.

Hence the Proof.

Lemma 9. The predator population will be decreased if the anti-predator behavior (η) of the adult prey population increased.

The proof of this lemma is obvious.

6. GLOBAL STABILITY ANALYSIS

The ability of an ecological unit to withstand great disturbances without being greatly affected is called Global Stability. Since in a natural system all parametric values will be changed in time to time hence, the stability of the system will also be changed due to the changes of parametric values. So, the evaluation of global stability condition (Li and Muldowney [10]) of the positive equilibrium is necessary.

Theorem.3 The proposed system (1) will be globally asymptotically stable around its positive equilibrium point, provided that

$$\frac{\mu\beta_1}{1+k_1+k_2}+\frac{\mu_1\beta_2}{1+k_3+k_4}-d_1-\eta\xi > L_1'+L_3'+L_4'$$
$$\frac{\mu\beta_1}{1+k_1+k_2}+\frac{\mu_1\beta_2}{1+k_3+k_4}-d_1-\eta\xi > L_1'+L_3'+L_2'$$
$$\frac{\mu\beta_1}{1+k_1+k_2}+\frac{\mu_1\beta_2}{1+k_3+k_4}-d_1-\eta\xi > L_1'+\beta+L_6'+L_7'$$
$$\frac{\mu\beta_1}{1+k_1+k_2}+\frac{\mu_1\beta_2}{1+k_3+k_4}-d_1-\eta\xi > L_3'+L_5'+L_7'$$

where $L_1',.....,L_7'$ and ξ have been mentioned inside the proof.

Proof. The autonomous system (1) can be written in the following form

$$\frac{dX}{dt}=f(X) \tag{17}$$

where

$$f(X)=\begin{pmatrix} rx(1-\frac{x}{k})-\beta x-\frac{\beta_1 xz^2}{z^2+k_1xz+k_2x^2} \\ \beta x-dy-\frac{\beta_2 yz^2}{z^2+k_3yz+k_4y^2} \\ \frac{\mu\beta_1 xz^2}{z^2+k_1xz+k_2x^2}+\frac{\mu_1\beta_2 yz^2}{z^2+k_3yz+k_4y^2}-d_1z-\eta yz \end{pmatrix}, X=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Then the jacobian matrix for the system (17) obtained from equation (13) is given by

$$J(x,y,z)=\begin{pmatrix} L_1 & 0 & L_2 \\ \beta & L_3 & L_4 \\ L_5 & L_6 & L_7 \end{pmatrix}$$

where $L_1=r-\frac{2rx}{k}-\beta-\frac{\beta_1 z^2}{z^2+k_1xz+k_2x^2}+\frac{\beta_1 xz^2(k_1z+2k_2x)}{(z^2+k_1xz+k_2x^2)^2}$, $L_2=\frac{\beta_1 xz^2(2z+k_1x)}{(z^2+k_1xz+k_2x^2)^2}-\frac{2\beta_1 xz}{z^2+k_1xz+k_2x^2}$, $L_3=-d-\frac{\beta_2 z^2}{z^2+k_3yz+k_4y^2}+\frac{\beta_2 yz^2(2k_4y+k_3z)}{(z^2+k_3yz+k_4y^2)^2}$, $L_4=\frac{\beta_2 yz^2(2z+k_3y)}{(z^2+k_3yz+k_4y^2)^2}-\frac{2\beta_2 yz}{z^2+k_3yz+k_4y^2}$, $L_5=\frac{\mu\beta_1 z^2}{z^2+k_1xz+k_2x^2}-\frac{\mu\beta_1 xz^2(k_1z+2k_2x)}{(z^2+k_1xz+k_2x^2)^2}$, $L_6=\frac{\mu_1\beta_2 z^2}{z^2+k_3yz+k_4y^2}-\frac{\mu_1\beta_2 yz^2(2k_4y+k_3z)}{(z^2+k_3yz+k_4y^2)^2}-\eta z$, $L_7=\frac{2\mu\beta_1 xz}{z^2+k_1xz+k_2x^2}-\frac{\mu\beta_1 xz^2(2z+k_1x)}{(z^2+k_1xz+k_2x^2)^2}+\frac{2\mu_1\beta_2 yz}{z^2+k_3yz+k_4y^2}-\frac{\mu_1\beta_2 yz^2(2z+k_3y)}{(z^2+k_3yz+k_4y^2)^2}-d_1-\eta y.$

If $J^{[2]}$ be the second additive compound matrix of J can be expressed as

$$J^{[2]}=\begin{pmatrix} L_1+L_3 & L_4 & -L_2 \\ L_6 & L_1+L_7 & 0 \\ -L_5 & \beta & L_3+L_7 \end{pmatrix}$$

Now, let us consider a function $M(X)$ in such a way that

$$M=diag\{\frac{x}{z},\frac{x}{z},\frac{x}{z}\} \text{ and } M^{-1}=diag\{\frac{z}{x},\frac{z}{x},\frac{z}{x}\}$$

Again, we define

$$M_f=\frac{dM}{dx}=diag\{\frac{\dot{x}}{z}-\frac{\dot{x}}{z^2}\dot{z},\frac{\dot{x}}{z}-\frac{\dot{x}}{z^2}\dot{z},\frac{\dot{x}}{z}-\frac{\dot{x}}{z^2}\dot{z}\}$$
$$M_fM^{-1}=diag\{\frac{\dot{x}}{x}-\frac{\dot{z}}{z},\frac{\dot{x}}{x}-\frac{\dot{z}}{z},\frac{\dot{x}}{x}-\frac{\dot{z}}{z}\}$$
$$MJ^{[2]}M^{-1}=J^{[2]}$$

We have

$$B=M_fM^{-1}+MJ^{[2]}M^{-1}$$
$$=\begin{pmatrix} \frac{\dot{x}}{x}-\frac{\dot{z}}{z}+L_1+L_3 & L_4 & -L_2 \\ L_6 & \frac{\dot{x}}{x}-\frac{\dot{z}}{z}+L_1+L_7 & 0 \\ -L_5 & \beta & \frac{\dot{x}}{x}-\frac{\dot{z}}{z}+L_3+L_7 \end{pmatrix}$$

where

$$B_{11}=\frac{\dot{x}}{x}-\frac{\dot{z}}{z}+L_1+L_3, B_{12}=\begin{pmatrix} L_4 & -L_2 \end{pmatrix}$$
$$B_{21}=\begin{pmatrix} L_6 \\ -L_5 \end{pmatrix}$$

$$B_{22}=\begin{pmatrix} \frac{\dot{x}}{x}-\frac{\dot{z}}{z}+L_1+L_7 & 0 \\ \beta & \frac{\dot{x}}{x}-\frac{\dot{z}}{z}+L_3+L_7 \end{pmatrix}$$

Now,

$$\Gamma_1(B_{11})=\frac{\dot{x}}{x}-\frac{\dot{z}}{z}+L_1+L_3, |B_{12}|$$
$$=max\{L_4,|-L_2|\}, |B_{21}|=max\{L_6,|-L_5|\}$$
$$\Gamma_1(B_{22})=\frac{\dot{x}}{x}-\frac{\dot{z}}{z}+L_7+max\{L_1+\beta,L_3\}$$

Again, from the third equation of system (1), it is obtained that

$$\frac{\dot{z}}{z}=\frac{\mu\beta_1 xz}{z^2+k_1xz+k_2x^2}+\frac{\mu_1\beta_2 yz}{z^2+k_3yz+k_4y^2}-d_1-\eta y \tag{18}$$

Here,

$$p_1=\Gamma_1(B_{11})+|B_{12}|$$
$$=\frac{\dot{x}}{x}-\frac{\dot{z}}{z}+L_1+L_3+max\{L_4,|-L_2|\}$$
$$=\frac{\dot{x}}{x}+max\{L_1+L_3+L_4-\frac{\dot{z}}{z},L_1+L_3+|-L_2|-\frac{\dot{z}}{z}\}$$
$$p_2=\Gamma_1(B_{22})+|B_{21}|$$
$$=\frac{\dot{x}}{x}-\frac{\dot{z}}{z}+L_7+max\{L_1+\beta,L_3\}+max\{L_6,|-L_5|\}$$
$$=\frac{\dot{x}}{x}+max\{L_1+\beta+L_6+L_7-\frac{\dot{z}}{z},L_3+|-L_5|+L_7-\frac{\dot{z}}{z}\}$$

Now,

$$\Gamma(B) \le max\{p_1, p_2\}$$
$$\Gamma(B) \le \frac{\dot{x}}{x} + max\{L_1 + L_3 + L_4 - \frac{\dot{z}}{z}, L_1 + L_3 + |-L_2| - \frac{\dot{z}}{z},$$
$$L_1 + \beta + L_6 + L_7 - \frac{\dot{z}}{z}, L_3 + |-L_5| + L_7 - \frac{\dot{z}}{z}\}$$
$$\Gamma(B) \le \frac{\dot{x}}{x} - min\{\frac{\dot{z}}{z} - L_1 - L_3 - L_4, \frac{\dot{z}}{z} - L_1 - L_3 - |-L_2|,$$
$$\frac{\dot{z}}{z} - L_1 - \beta - L_6 - L_7, \frac{\dot{z}}{z} - L_3 - |-L_5| - L_7\}$$

the above equation can be written as

$$\Gamma(B) \le \frac{\dot{x}}{x} - w \tag{19}$$

where the expression of w ca be found by using the value of equation (18) as

$$w = min\{\frac{\mu\beta_1}{1+k_1+k_2} + \frac{\mu_1\beta_2}{1+k_3+k_4} - d_1 - \eta\xi - L_1' - L_3' - L_4',$$
$$\frac{\mu\beta_1}{1+k_1+k_2} + \frac{\mu_1\beta_2}{1+k_3+k_4} - d_1 - \eta\xi - L_1' - L_3' - L_2',$$
$$\frac{\mu\beta_1}{1+k_1+k_2} + \frac{\mu_1\beta_2}{1+k_3+k_4} - d_1 - \eta\xi - L_1' - \beta - L_6' - L_7',$$
$$\frac{\mu\beta_1}{1+k_1+k_2} + \frac{\mu_1\beta_2}{1+k_3+k_4} - d_1 - \eta\xi - L_3' - L_5' - L_7'\}$$

and

$$L_1' = r - \frac{2r\xi}{k} - \beta - \frac{\beta_1}{1+k_1+k_2} + \frac{\beta_1(k_1+2k_2)}{(1+k_1+k_2)^2}$$
$$L_2' = |-L_2| = |\frac{2\beta_1}{1+k_1+k_2} - \frac{\beta_1(2+k_1)}{(1+k_1+k_2)^2}|$$
$$L_3' = -d - \frac{\beta_2}{1+k_3+k_4} + \frac{\beta_2(2k_4+k_3)}{(1+k_3+k_4)^2}$$
$$L_4' = \frac{\beta_2(2+k_3)}{(1+k_3+k_4)^2} - \frac{2\beta_2}{1+k_3+k_4}$$
$$L_5' = |-L_5| = |\frac{\mu\beta_1(k_1+2k_2)}{(1+k_1+k_2)^2} - \frac{\mu\beta_1}{1+k_1+k_2}|$$
$$L_6' = \frac{\mu_1\beta_2}{1+k_3+k_4} - \frac{\mu_1\beta_2(2k_4+k_3)}{(1+k_3+k_4)^2} - \eta\xi$$
$$L_7' = \frac{2\mu\beta_1}{1+k_1+k_2} - \frac{\mu\beta_1(2+k_1)}{(1+k_1+k_2)^2} + \frac{2\mu_1\beta_2}{1+k_3+k_4}$$
$$-\frac{\mu_1\beta_2(2+k_3)}{(1+k_3+k_4)^2} - d_1 - \eta\xi$$

also

$$\xi = min\{x(t), y(t), z(t)\}.$$

Then integrating equation (19) from 0 to t it is obtained that

$$\int_0^t \Gamma(B)ds \le \int_0^t \frac{\dot{x}}{x}dt - w\int_0^t dt$$
$$\int_0^t \Gamma(B)ds \le log\frac{x(t)}{x(0)} - wt$$
$$i.e., \frac{1}{t}\int_0^t \Gamma(B)ds \le \frac{1}{t}log\frac{x(t)}{x(0)} - w$$
$$i.e., \lim_{t\to\infty} \sup\sup \frac{1}{t}\int_0^t \mu(B)ds < -w < 0, \text{ provided that } w > 0$$

Now, the value of w will be positive if

$$\frac{\mu\beta_1}{1+k_1+k_2} + \frac{\mu_1\beta_2}{1+k_3+k_4} - d_1 - \eta\xi > L_1' + L_3' + L_4'$$
$$\frac{\mu\beta_1}{1+k_1+k_2} + \frac{\mu_1\beta_2}{1+k_3+k_4} - d_1 - \eta\xi > L_1' + L_3' + L_2'$$
$$\frac{\mu\beta_1}{1+k_1+k_2} + \frac{\mu_1\beta_2}{1+k_3+k_4} - d_1 - \eta\xi > L_1' + \beta + L_6' + L_7'$$
$$\frac{\mu\beta_1}{1+k_1+k_2} + \frac{\mu_1\beta_2}{1+k_3+k_4} - d_1 - \eta\xi > L_3' + L_5' + L_7'$$

Hence the theorem.

7. NUMERICAL SIMULATIONS

In this section, the dynamical behaviour of the proposed model (1) has been discussed numerically using MATLAB. Due to unavailability of real data of all parameters associated with the model, the hypothetical values of different parameters have been considered as follows: $r = 0.05, k = 15, \beta = 0.1, \beta_1 = 0.2, k_1 = 0.0001, k_2 = 0.0003, d = 0.01, \beta_2 = 0.2, k_3 = 0.002, k_4 = 0.003, \mu = 0.9, \mu_1 = 0.85, d_1 = 0.015, \eta = 0.01$. Now, for this data set it is observed that the system (1) is locally asymptotically stable around the equilibrium point $E_0(0,0,0)$ since $r < \beta$. This is also shown in Fig.1 graphically.

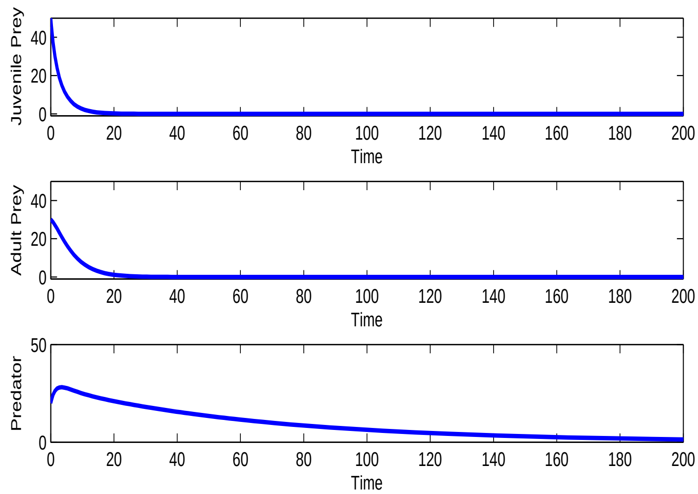

Fig.1 representation of stability around E_0.

Again, we consider another set of parametric values such as $r = 5, k = 15.0, \beta = 0.0, \beta_1 = 0.01, k_1 = 0.1, k_2 = 0.1, d = 0.1, \beta_2 = 0.01, k_3 = 0.1, k_4 = 0.1, \mu = 0.4, \mu_1 = 0.45, d_1 = 0.15, \eta = 0.01$. Now, for this data set it is observed that the system (1) is locally asymptotically stable around the equilibrium point $E_1(15,0,0)$ since $\beta = 0$. This is also shown in Fig.2 graphically.

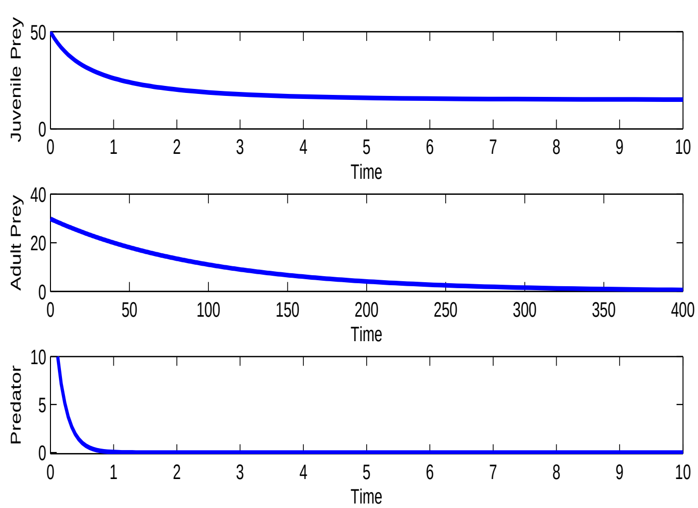

Fig.2 representation of stability around E_1.

Again, we also consider another set of parametric values such as $r = 5, k = 15.0, \beta = 0.1, \beta_1 = 0.02, k_1 = 0.1, k_2 = 0.1, d = 0.1, \beta_2 = 0.02, k_3 = 0.2, k_4 = 0.3, \mu = 0.4, \mu_1 = 0.45, d_1 = 0.15, \eta = 0.01$.

Now, for this data set it is observed that the system (1) is locally asymptotically stable around the equilibrium point $E_2 = (14.7, 14.7, 0)$ since $r > \beta$ according to **Lemma 6**. This is also shown in Fig.3 graphically.

Now, we consider the set of parametric values: such as $r = 5, k = 15.0, \beta = 0.0, \beta_1 = 0.01, k_1 = 0.01, k_2 = 0.01, d = 0.1, \beta_2 = 0.01, k_3 = 0.1, k_4 = 0.1, \mu = 0.4, \mu_1 = 0.45, d_1 = 0.15, \eta = 0.01$.

Now, for this data set it is observed that the system (1) is locally asymptotically stable around the equilibrium point $E_3 = (14.99, 0, 0.39)$ since $c_1 = 5.2633 > 0, c_3 = 0.0826 > 0$ and $c_1c_2 - c_3 = 6.8512 > 0$ according to **Lemma 7.**. This is also shown in Fig.4 graphically.

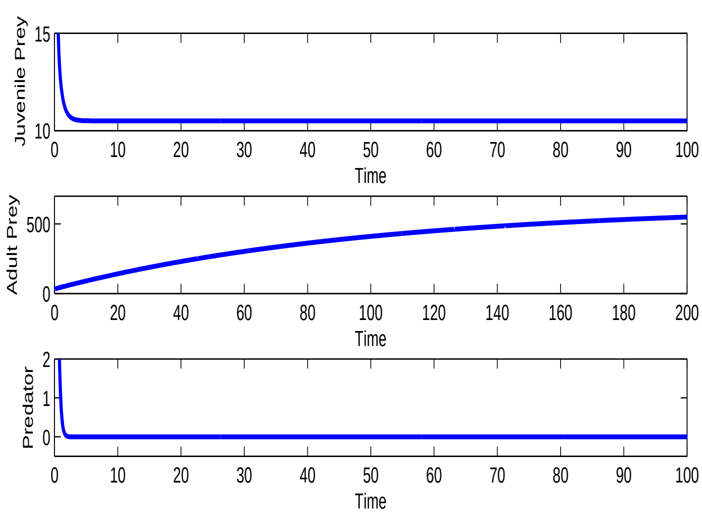

Fig.3 representation of stability around E_2.

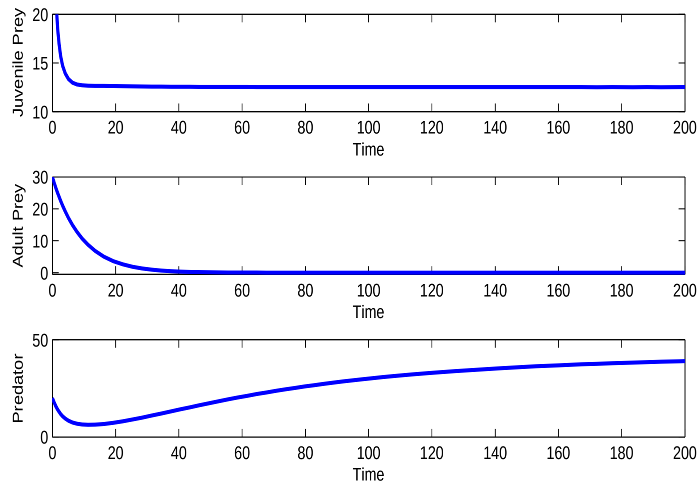

Fig.4 representation of stability around E_3.

For the same set of parametric values we draw the equilibrium curve for the equilibrium point $(x_1^*, 0, z_1^*)$. The stable and unstable equilibrium points have been shown separately in Fig.5.

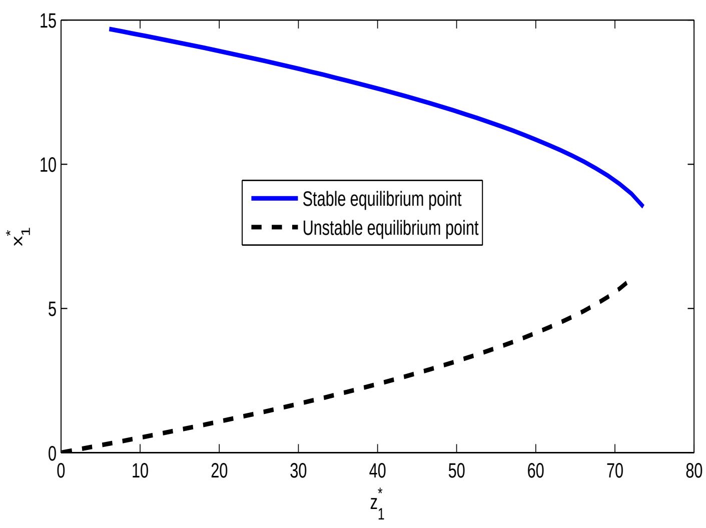

Fig.5 curve of equilibrium points of E_3.

Again, we also consider another set of parametric values: such as $r = 0.9, k = 15, \beta = 0.1, \beta_1 = 0.2, k_1 = 0.01, k_2 = 0.03, d = 0.01, \beta_2 = 0.2, k_3 = 0.002, k_4 = 0.003, \mu = 0.9, \mu_1 = 0.85, d_1 = 0.015, \eta = 0.01$.

Now, for this parametric values it is seen that $P_1 = 0.0762 > 0, P_2 = 0.8050 > 0, P_3 = 2.6472 > 0$. Hence, according to **Lemma 5** there exists a positive equilibrium point which is $E_4 = (10, 4.722, 41.56)$. Again, for this data set it is observed that the system (1) is locally asymptotically stable around the equilibrium point $E_4 = (10, 4.722, 41.56)$ since $b_1 = 0.8704 > 0, b_3 = 0.0078 > 0, b_1 b_2 - b_3 = 0.1454 > 0$ according to **Lemma 8**. This is also shown graphically in Fig.6. Also, for this set of parametric values we have

$$
\begin{aligned}
L_1' &= r - \frac{2r\xi}{k} - \beta - \frac{\beta_1}{1+k_1+k_2} + \frac{\beta_1(k_1+2k_2)}{(1+k_1+k_2)^2} = 0.0540 \\
L_2' &= |-L_2| = |\frac{2\beta_1}{1+k_1+k_2} - \frac{\beta_1(2+k_1)}{(1+k_1+k_2)^2}| = 0.0129 \\
L_3' &= -d - \frac{\beta_2}{1+k_3+k_4} + \frac{\beta_2(2k_4+k_3)}{(1+k_3+k_4)^2} = -0.2074 \\
L_4' &= \frac{\beta_2(2+k_3)}{(1+k_3+k_4)^2} - \frac{2\beta_2}{1+k_3+k_4} = -0.0016 \\
L_5' &= |-L_5| = |\frac{\mu\beta_1(k_1+2k_2)}{(1+k_1+k_2)^2} - \frac{\mu\beta_1}{1+k_1+k_2}| = 0.1614 \\
L_6' &= \frac{\mu_1\beta_2}{1+k_3+k_4} - \frac{\mu_1\beta_2(2k_4+k_3)}{(1+k_3+k_4)^2} - \eta\xi = 0.1206
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
L_7' = \frac{2\mu\beta_1}{1+k_1+k_2} - \frac{\mu\beta_1(2+k_1)}{(1+k_1+k_2)^2} + \frac{2\mu_1\beta_2}{1+k_3+k_4} - \frac{\mu_1\beta_2(2+k_3)}{(1+k_3+k_4)^2} \\
-d_1 - \eta\xi = -0.0492 \\
\frac{\mu\beta_1}{1+k_1+k_2} + \frac{\mu_1\beta_2}{1+k_3+k_4} - d_1 - \eta\xi = 0.2800 \text{ and } \xi = 4.722
\end{aligned}
$$

Then, all conditions of **Theorem 3** have been satisfied. Therefore, it is concluded that our proposed system (1) is globally asymptotically stable around the positive equilibrium point $E_4 = (10, 4.722, 41.56)$ for this data set.

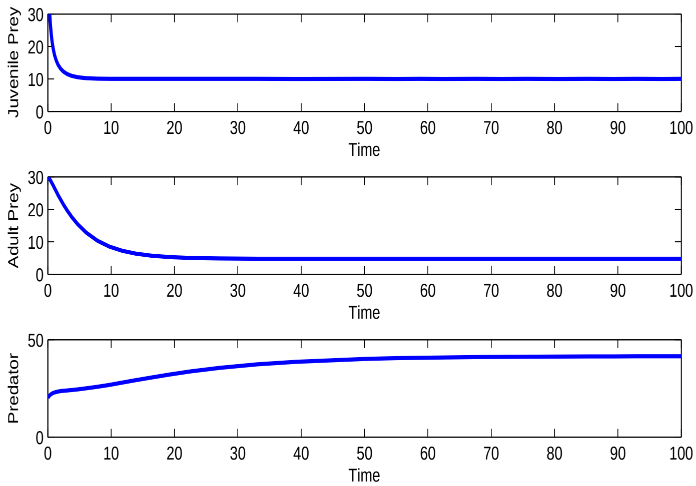

Fig.6 representation of stability around E_4.

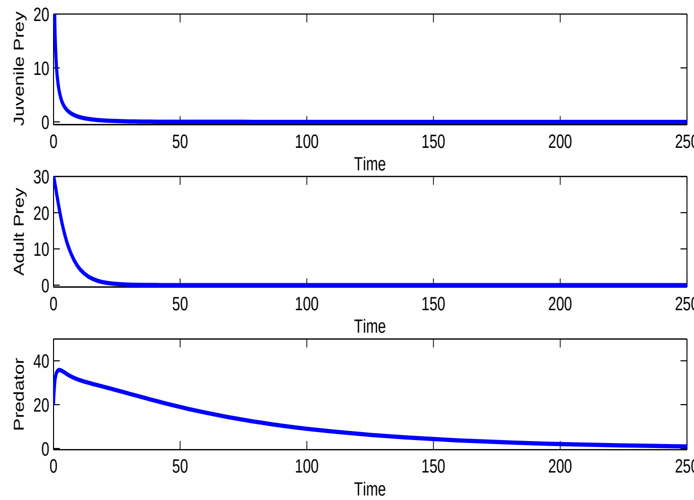

Fig.7 extinction of populations when $\beta_1 > (r-\beta)$.

Now, taking the same set of parametric values which have been used in Fig.6 except $\beta_1 = 0.9, k_1 = k_2 = k_3 = k_4 = 0$, Fig.7 has been drawn. From this figure, it is observed that all three populations are going to extinction with respect to time. That is, if (i) the handling time of the predator to consume the both juvenile prey and adult prey are zero and (ii) there does not exist no competition between preys i.e., if the consumption rate of a predator to an juvenile prey is grater than the difference of the intrinsic growth rate of the juvenile prey and the transmission rate from the juvenile prey to the adult prey, then both the prey and predator populations extinct after sometimes. This has been also proved analytically in **Theorem 2**.

Finally, the effects of anti-predator behaviour of adult prey have been shown in Fig.8 taking the parametric values which have been used in Fig.6. From, this figure it is observed that when the value of anti-predator behavior η increases, then the predator population gradually decreases and ultimately it becomes extinct.

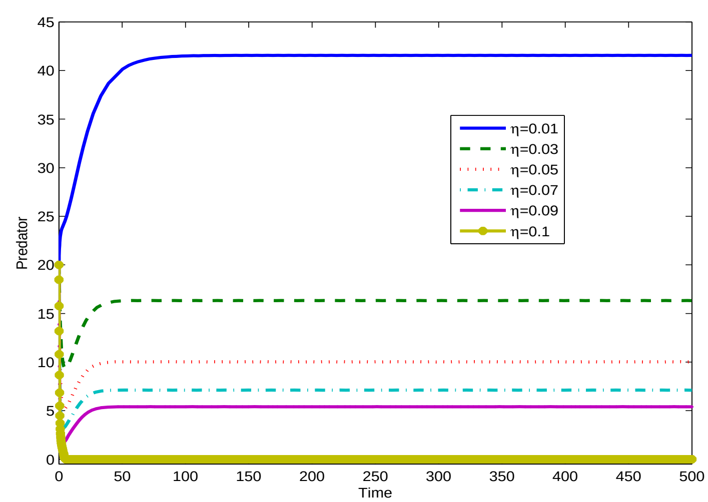

Fig.8 sensitivity analysis of the anti-predator behavior η.

8. CONCLUSION

In this paper, a predator-prey model has been considered. In this model it has been introduced that the adult prey has anti-predator behaviour and consequently a predator attacks more juvenile prey than adult prey. That is, here the prey population has been divided into two subpopulations such as (i) juvenile prey (ii) adult prey. Here, a Holling type-IV functional response has been used on the basis of ratio-dependency of prey and predator. After deriving the existence condition and boundedness of solutions of the model, the local stability of the system around the different equilibrium points and global stability of the system around interior equilibrium point have been discussed theoretically and numerically. Here a condition has been derived when the prey and predator both population extinct. We observe that if the growth rate of juvenile prey is less than the transmission rate (β) of adult prey from the juvenile prey population then all the three populations go to extinction. It is also observed that if the consumption rate of a predator to an juvenile prey is grater than the difference of the intrinsic growth rate of the juvenile prey and the transmission rate from the juvenile prey to the adult prey then the three populations also go to extinction. At last it is observed that if the anti-predator behaviour of adult prey increases then the predator population may be extinct.

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