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A Common Fixed Point Result for Multi-Valued Mappings in Spherically Complete Ultrametric Spaces

Abdelbasset Felhi[1,*]

[1] *King Faisal University, College of Sciences. Department of Mathematics and Statistics, Al-hassa, Saudi Arabia*

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In this paper, we apply the strong contractive type mappings on the results of Rhodes [10] and prove a common fixed point theorem for a single-valued and the multi-valued mappings in spherically complete ultrametric spaces. The presented results unify, extend and improve several results in the related literature.

Keywords: ultrametric space, spherically complete, Multi-valued maps, fixed point.

1. INTRODUCTION AND PRELIMINARIES

In 1978, A. C. M. van Roovij [1] introduced the concept of ultrametric spaces. Since then, several fixed point and common fixed point theorems in the framework of ultrametric spaces have been investigated in [2]-[9]. In 1977, Rhodes [10] listed contractive type mappings which were generalizations of Banach contraction principle.

Now, we give some basic definitions and results which are used throughout the paper.

Definition 1.1 *[2] Let (X,d) be a metric space. If the metric d satisfies the strong triangle inequality:*

$$d(x,y) \leq \max\{d(x,y),d(y,z)\} \quad \text{for all } x,y,z \in X,$$

it said to be ultrametric on X. The pair (X,d) is said to be an ultrametric space.

Example 1.2 *The discrete metric d defined on $X \neq \emptyset$ by*

$$d(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$$

is an ultrametric.

Definition 1.3 *[2] An ultra metric space (X,d) is said to be spherically complete if every shrinking collection of balls in X has a nonempty intersection.*

Definition 1.4 *An element $x \in X$ is called a coincidence point of $S : X \to X$ and $T : X \to 2^X_c$ (where 2^X_c is the space of all nonempty compact subsets in X) if $Sx \in Tx$.*

Definition 1.5 *Let $S : X \to X$ and $T : X \to 2^X_c$. The mappings S and T are called coincidentally commuting at $x \in X$ if $STx \subseteq TSx$ whenever $Sx \in Tx$.*

Theorem 1.6 *(Zorn's lemma) Let S be a partially ordered set. If every totally ordered subset of S has an upper bound, then S contains a maximal element.*

In 2002, Lj. Gajic [3] proved the following result.

Theorem 1.7 *([3]) Let (X,d) be a spherically complete ultrametric space. If $T : X \to 2^X_c$ is such that for any $x,y \in X$, $x \neq y$,*

$$H(Tx,Ty) < \max\{d(x,y),d(x,Tx),d(y,Ty)\},$$

then T has a fixed point, that is, there exists $x \in X$ such that $x \in Tx$, where H is the Hausdorff metric induced by the metric d.

* afelhi@kfu.edu.sa

In this paper, we establish a unique common fixed point theorem for a single-valued and the multi-valued maps involving some strong contractive type mappings in spherically complete ultrametric spaces.

2. MAIN RESULTS

In this section, we apply strong contractive type mappings on the results of Rhoades [10] and established some new fixed point results in ultrametric spaces for multi-valued maps. Let us prove our main result.

Theorem 2.1 *Let (X,d) be an ultrametric space. Let $S : X \to X$ and $T : X \to 2^X_c$ be maps satisfying*

$$H(Tx,Ty) < \max\{d(Sx,Sy), d(Sx,Tx), d(Sy,Ty), d(Sx,Ty), d(Sy,Tx)\} \tag{2.1}$$

for all $x, y \in X$ such that $x \neq y$.
Suppose that

(*i*) *SX is spherically complete.*

Then there exists $w \in X$ such that $Sw \in Tw$.
Assume in addition that

(*ii*) *S and T are coincidentally commuting at w;*

(*iii*) *$d(Sx,Sy) \leq d(y,Tx)$ for all $x, y \in X$.*

Then Sw is the unique common fixed point of S and T, that is, $S(Sw) = Sw \in T(Sw)$.

Proof. Assume that $d(Sx,Tx) = \inf_{z \in Tx} d(Sx,z) > 0$ for all $x \in X$.

Let $B_a = B[Sa, d(Sa,Ta)] \cap SX$ denote the closed ball centred at Sa with radius $d(Sa,Ta) > 0$ for all $a \in X$ and let F be the collection of these balls. We define on F the following partial order

$$B_a \preceq B_b \Leftrightarrow B_b \subseteq B_a.$$

Let F_1 be a totally ordered subfamily of F. We shall prove that F_1 has an upper bound. By condition (i), SX is spherically complete, it follows that

$$\bigcap_{B_a \in F_1} B_a = B \neq \emptyset.$$

Let $Sb \in B$. This implies that $Sb \in B_a$, as $B_a \in F_1$. So $d(Sb,Sa) \leq d(Sa,Ta)$. Since Ta is nonempty compact set, then there exists $u \in Ta$ such that $d(Sa,u) = d(Sa,Ta)$. From (2.1) and by the strong triangle inequality, we get

$$\begin{aligned} d(Sb,Tb) &\leq \max\{d(Sb,Sa), d(Sa,u), d(u,Tb)\} \\ &\leq \max\{d(Sa,Ta), H(Ta,Tb)\} \\ &< \max\{d(Sa,Ta), d(Sa,Sb), d(Sa,Ta), d(Sb,Tb), d(Sa,Tb), d(Sb,Ta)\} \\ &= \max\{d(Sa,Ta), d(Sb,Tb), d(Sa,Tb), d(Sb,Ta)\}. \end{aligned}$$

As $d(Sa,Tb) \leq \max\{d(Sa,Sb), d(Sb,Tb)\}$ and $d(Sb,Ta) \leq \max\{d(Sb,Sa), d(Sa,Ta)\}$, then

$$d(Sb,Tb) < \max\{d(Sa,Ta), d(Sb,Tb)\}.$$

Necessarily, we have $d(Sb,Tb) < d(Sa,Ta)$.
For $x \in B_b$, we have

$$d(Sb,x) \leq d(Sb,Tb) < d(Sa,Ta).$$

Then

$$d(Sa,x) \leq \max\{d(Sa,Sb), d(Sb,x)\} \leq d(Sa,Ta).$$

It follows that $x \in B_a$ and so $B_b \subseteq B_a$. Thus $B_a \preceq B_b$ for all $B_a \in F_1$. Hence B_b is an upper bound in F for the family F_1. By Zorn's lemma, there exists a maximal element in F, say B_w. We claim that $Sw \in Tw$. We argue by contradiction, that is, $Sw \notin Tw$. Since Tw is a nonempty compact set, there exists $Sv \in Tw$ such that $d(Sv,Sw) = d(Sw,Tw)$ and $Sv \neq Sw$. We shall prove that $B_v \subseteq B_w$.
We have

$$\begin{aligned} d(Sv,Tv) &\leq H(Tw,Tv) \\ &< \max\{d(Sw,Sv), d(Sw,Tw), d(Sv,Tv), d(Sw,Tv), d(Sv,Tw)\} \\ &< \max\{d(Sw,Tw), d(Sv,Tv), d(Sw,Sv), d(Sv,Tv), d(Sv,Sw), d(Sw,Tw)\} \\ &= \max\{d(Sw,Tw), d(Sv,Tv)\}. \end{aligned}$$

Then $d(Sv,Tv) < d(Sw,Tw)$. Now, for $x \in B_v$, we have

$$d(Sv,x) \leq d(Sv,Tv) < d(Sw,Tw).$$

It follows that

$$d(Sw,x) \leq \max\{d(Sw,Sv), d(Sv,x)\} = d(Sw,Tw).$$

Hence $x \in B_w$ and so $B_v \subseteq B_w$. Moreover, $Sw \in B_w$ but $Sw \in B_v$, because $d(Sv,Sw) = d(Sw,Tw) > d(Sv,Tv)$. Then $Bv \subsetneqq Bw$, which is a contradiction to the maximality of B_w. Hence

$Sw \in Tw$. Let $z = Sw$. We claim that z is a common fixed point of S and T.

By condition (iv), we have

$$d(z, Sz) = d(Sw, S(Sw)) \leq d(Sw, Tw) = 0,$$

because $Sw \in Tw$, which implies that $z = Sz$. Further, as $Sw \in Tw$, by condition (ii), we get $S(Sw) \in STw \subseteq TSw$. Then, $z = Sz \in Tz$. Hence, z is a common fixed point of S and T.

Let z' another common fixed point of S and T. Suppose that $z \neq z'$. Using the condition (iii), from (2.1), we have

$$\begin{aligned} 0 < d(z,z') &= d(Sz,Sz') \\ &\leq d(z',Tz) \\ &\leq H(Tz',Tz) \\ &< \max\{d(Sz,Sz'),d(Sz,Tz),d(Sz',Tz'),d(Sz,Tz'),d(Sz',Tz)\} \\ &= \max\{d(z,z'),d(z,Tz'),d(z',Tz)\} \\ &\leq \max\{d(z,z'),d(z,z'),d(z',Tz'),d(z',z),d(z,Tz)\} \\ &= d(z,z'), \end{aligned}$$

which is a contradiction. Hence $z = z'$.

If there exists $x \in X$ such that $d(Sx, Tx) = 0$, then $Sx \in Tx$. Similarly, we prove that Sx is the unique common fixed point of S and T and this completes the proof. ■

Corollary 2.2 *Let (X,d) be a spherically complete ultrametric space. Let $T : X \to 2^X_c$ be a multi-valued map satisfying*

$$H(Tx,Ty) < \max\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}$$

for all $x,y \in X$ such that $x \neq y$. Then T has a fixed point. Assume in addition that $d(x,y) \leq d(y,Tx)$ for all $x,y \in X$. Then, the fixed point of T is unique.

Corollary 2.3 *Let (X,d) be an ultrametric space. Let $S : X \to X$ and $T : X \to X$ be maps satisfying*

$$d(Tx,Ty) < \max\{d(Sx,Sy),d(Sx,Tx),d(Sy,Ty),d(Sx,Ty),d(Sy,Tx)\}$$

for all $x,y \in X$, $x \neq y$.
Suppose that

(i) SX is spherically complete.

Then there exists $w \in X$ such that $Sw = Tw$.

Assume in addition that

(ii) S and T are coincidentally commuting at w.

Then Sw is the unique common fixed point of S and T.

Corollary 2.4 *Let (X,d) be a spherically complete ultrametric space. Let $T : X \to X$ be a map satisfying*

$$d(Tx,Ty) < \max\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}$$

for all $x,y \in X$ such that $x \neq y$. Then T has a unique fixed point.

Corollary 2.5 *Let (X,d) be an ultrametric space. Let $S : X \to X$ and $T : X \to 2^X_c$ such that*

$$H(Tx,Ty) < \max\{d(Sx,Sy),d(Sx,Tx),d(Sy,Ty)\}$$

for all $x,y \in X$ such that $x \neq y$. Suppose that

(i) SX is spherically complete.

Then there exists $w \in X$ such that $Sw \in Tw$.

Assume in addition that

(ii) S and T are coincidentally commuting at w;

(iii) $d(Sx,Sy) \leq d(y,Tx)$ for all $x,y \in X$.

Then Sw is the unique common fixed point of S and T, that is, $S(Sw) = Sw \in T(Sw)$.

Corollary 2.6 *([3], Theorem) Let (X,d) be a spherically complete ultrametric space. If $T : X \to 2^X_c$ is such that for any $x,y \in X$, $x \neq y$,*

$$H(Tx,Ty) < \max\{d(x,y),d(x,Tx),d(y,Ty)\},$$

then T has a fixed point.

Corollary 2.7 *Let (X,d) be an ultrametric space. Let $S : X \to X$ and $T : X \to 2^X_c$ such that*

$$H(Tx,Ty) < \max\{d(Sx,Sy),d(Sx,Ty),d(Sy,Tx)\}$$

for all $x,y \in X$ such that $x \neq y$. Suppose that

(i) SX is spherically complete.

Then there exists $w \in X$ such that $Sw \in Tw$.

Assume in addition that

(ii) *S and T are coincidentally commuting at w;*

(iii) $d(Sx, Sy) \leq d(y, Tx)$ *for all* $x, y \in X$.

Then Sw is the unique common fixed point of S and T, that is, $S(Sw) = Sw \in T(Sw)$.

Corollary 2.8 *Let* (X, d) *be a spherically complete ultrametric space. If* $T : X \to 2^X_c$ *is such that for any* $x, y \in X$, $x \neq y$,

$$H(Tx, Ty) < \max\{d(x, y), d(y, Tx), d(x, Ty)\},$$

then T has a fixed point. Assume in addition that $d(x, y) \leq d(y, Tx)$ *for all* $x, y \in X$. *Then, the fixed point of T is unique.*

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