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Semi-invariant $\xi^{\perp}$- Submanifolds in Metric Geometry of Affinors

Mohd. Danish Siddiqi[1, *]

[1] *Department of Mathematics, College of Science, Jazan University, Jazan, Kingdom of Saudi Arabia*

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In this paper, we study semi-invariant $\xi^{\perp}$-submanifolds which endowed with a generalization of structured manifolds as the most general Riemannian metric g associated to an affinor F. We have also investigate the integrability conditions of both invariant and anti-invariant ditributions and characterizations for totally geodesic case.

Keywords: semi-invariant $\xi^{\perp}$- submanifolds, affinors, integrability Conditions, distributions.

1. INTRODUCTION

In the differential geometry the theory of submanifolds in spaces endowed with additional structures has been intensively studied and several important papers have been appeared in this field. Let us mention only few of them: a series of papers of B.Y. Chen (see, [5], [6], [7]), N. Papaghiuc (see [11] [12])and A. Bejancu studied semi-invariant submanifold for almost contact structure (see, [3] [4]) as well as almost complex geometry [2] which had a great impact on developing of the theory of submanifolds in which these ambient spaces, for example A. Bejancu in [1] also studied *QR*-submanifolds in quaternionic manifold and M. Barros et al. investigated *CR*-submanifolds in quaternionic manifolds [5]. In [8] C. Calin et al. studied the semi-invariant $\xi^{\perp}$ submanifolds of generalized quasi-Saskian manifold. Moreover, N. C. Chiriac and M. Crasmareanu studied semi-invariant submanifolds in metric geometry of affinors [9]. The purpose of the present paper is to investigate the semi-invariant $\xi^{\perp}$ -submanifolds in metric geometry of affinors.

* anallintegral@gmail.com, msiddiqi@jazanu.edu.sa

2. PRELIMINARIES

Let M be an n-dimensional manifold for which we denote by $C^{\infty}(M)$, the algebra of smooth functions on M and by TM the $C^{\infty}(M)$-module of smooth sections of the tangent bundle TM of M; let X, Y, Z,.... denote such vector fields. We use the same notation V for any other vector bundle V over M. Let $T_1^1(M)$ be also the $C^{\infty}(M)$-module of $TM \otimes T^*M$ i.e. the real space of tensor fields of $(1,1)$-type on M. Let consider a fixed $F \in T_1^1(M)$ usually called *affinor* [10] or *vector 1-form*; the remarkable affinior of every manifold is the kronecker tensor field $I = (\delta^i_j)$.
Fix $\mu \in +1, -1$. Let now g be a Riemannian metric on M.

Definition 1 *Let M be a* (g, F, μ)*-manifold and if*

$$g(FX,Y) + \mu g(X,FY) = 0, \tag{2.1}$$

then the geometry of the strucrure (M, g, F, μ) *is called affinor-metric geometry. If in particular,* F_x *is nondegenerate at any point* $x \in M$ *then we say that M is a nondegenerate* (g, F, μ)*-manifold otherwise, M is called degenerate* (g, F, μ)*-manifold. The relation (2.1) says that the g-adjoint of F is* $F = -\mu F^*$.

Now, let an n-dimensional submanifold N of M. Then the main object induced by the Levi-Civita connection $\bar{\nabla}$ of M on N are involved in the well known Gauss-Weingarten equations

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad X,Y \in TN, \tag{2.2}$$

$$\bar{\nabla}_X N = -A_V X + \nabla_X^{\perp} V, \quad N \in T^{\perp}N, \tag{2.3}$$

where $\bar{\nabla}$, ∇ and $\nabla^{\perp}$ are the Riemannian, induced Riemannian and induced normal connection in N, M and the normal bundle $T^{\perp}M$ of $\bar{M}$, respectively, and h is the second fundamental form related to A by the relation

$$g(h(X,Y),V) = g(A_V X,Y). \tag{2.4}$$

If $'h'$ vanishes identically on N, then N is called totally geodesic.

The aim of present paper is to investigate the semi-invariant $\xi^{\perp}$-submanifolds in metric geometry of affinors. More precisely, we suppose that the structure vector field ξ is orthogonal to the submanifold N.

3. SEMI-INVARIANT $\xi^{\perp}$- SUBMANIFOLD IN AFFINOR-METRIC GEOMETRY

Now, we consider a submanifold N of a (g,F,μ)-manifold M. Then g induces a Riemannain metric on N which we denote by the same symbol g. By following the definition given by Bejancu [1], we say that N is a semi-invariant $\xi^{\perp}$- submanifold if there exits two orthogonal distributions D and $D^{\perp}$ in TN such that $TN = D \oplus D^{\perp}$. If $D^{\perp} = 0$, then N is an invariant $\xi^{\perp}$- submanifold. The normal bundle can also be decomposed as $TN^{\perp} = FD^{\perp} \oplus \bar{D}$, where $F\bar{D} \subseteq \bar{D}$. If $\bar{D}$ contains ξ.

Definition 2 *N is a semi-invariant $\xi^{\perp}$-submanifold of M if there exists a distribution D on N satisfying the condition*

1. *D is F-invariant ie., $F(D_x) \subset D_x, \ \forall \ x \in N$*
2. *The complementary orthogonal distribution $D^{\perp}$ to D in TN is F-anti-invariant that is, $F(D^{\perp}) \subseteq T_x^{\perp}N, \ \forall \ x \in N,$*
3. *$F^2(D^{\perp})$ is a distribution on N.*

Some particular classes of semi-invariant $\xi^{\perp}$-submanifolds are defined as follows; Let p and q be the ranks of distributions D and $D^{\perp}$ respectively. If $q = 0$, i.e., $D^{\perp} = 0$ then we say that N is an F-invariant $\xi^{\perp}$-submanifold of M. If $p = 0$, that is, $D = 0$, we call N an F-antiinvariant $\xi^{\perp}$-submanifold of M. If $pq \neq 0$; then N is called a proper semi-invariant $\xi^{\perp}$-submanifold. We denote $\bar{D}$ the complementary orthogonal vector bundle to $FD^{\perp}$ in $T^{\perp}N$. If $\bar{D} = 0$, then we say that N is normal F-invariant, (respectively F-antiinvariant) if and only if

$$F(TN) \subset TN \quad (resp.\ F(TN) \subset T^{\perp}N).$$

N is normal F-semi-invariant $\xi^{\perp}$ if and only if $F(D^{\perp}) = T^{\perp}N$. Let N be a semi-invariant $\xi^{\perp}$-submanifold of a (g,F,μ)-manifold M. Denote P and Q the projection of TN on D and $D^{\perp}$ respectively, namely for any $X \in TN$

$$X = PX + QX. \tag{3.1}$$

Moreover, for any $X \in TN$ and $N \in TN^{\perp}$, we put

$$\phi X = tX + wX, \tag{3.2}$$

$$\phi X = BX + CX \tag{3.3}$$

with $tX \in D$, $BN \in TN$ and $wX, CN \in TN^{\perp}$. We also consider, for $X \in TN$, the decomposition

$$FX = \alpha X + \beta X, \ \ \alpha X \in D, \ \ \beta X \in TN^{N}, \tag{3.4}$$

where we put :

$$\alpha = F \circ P, \beta = F \circ Q. \tag{3.5}$$

Thus α is a tensor field of $(1,1)$-type on N while β is a $F(D^{\perp})$-valued vector 1-form on N. Thus we derive:

Proposition 1 *Let N be a semi-invariant $\xi^{\perp}$-submanifold of a (g,F,μ)-manifold M. Then*

1. *N is a (g,α,μ)-manifold.*
2. *$F^2(D^{\perp})$ is a vector subbundle of $D^{\perp}$.*
3. *The vector bundle $\bar{D}$ is F-invariant, i.e., for all $x \in N$ we have $F(\bar{D}) \subset \bar{D}_x$. Hence $\bar{D}$ contains ξ.*

Proof. *(1) By the definition, g is a Riemannian metric on N and α is a tensor field of type $(1,1)$ on N. Now from Definition (3.1) and using (2.1) for F we obtain for $X,Y \in TN$*

$$g(\alpha X,Y) = g(FPX,Y) = g(FPX,PY) = -\mu g(PX,FPY)$$

$$= \mu g(X,FPY) = -\mu g(X,\alpha Y).$$

(2) Take $X \in D$ and $Y \in D^{\perp}$ in Definition(3.1) part 2, $g(X,F^2Y) = -\mu g(FX,FY) = 0$ since $FX \in D$ and $FY \in T^{\perp}N$. Hence $F^2(D^{\perp})$ is orthogonal to D and again by the part (3) of Definition (3.1) we deduce that $F^2(D^{\perp})$ is a vector subbundle of $D^{\perp}$.

(3) Take $X \in TN$, $Y \in D^{\perp}$ and $V \in \bar{D}$. Then we obtain

$$g(FV,X) = -\mu g(V,FX) = -\mu g(V,\alpha X + \beta X) = 0$$

and

$$g(FV,FY) = -\mu g(V,F^2Y) = 0.$$

Since $\alpha X \in D$, $\beta X \in FD^{\perp}$ and $F^2Y \in D^{\perp}$. Thus $F\bar{D}$ is orthgonal to $TN \oplus FD^{\perp}$, that is $F\bar{D}$ is a vector subbundle of $\bar{D}$ which contains ξ. This complete the proof of the proposition. ■

Corollary 2 *Let N be a semi-invariant $\xi^{\perp}$-submanifold of a (g,F,μ)-manifold M. Then*

1. *the above distribution satisfy:*

$$F(D) = D, \quad F^2(D^{\perp}) = D^{\perp},\ F(\bar{D}) = \bar{D}. \tag{3.6}$$

2. *if $\mu = +1$, then $D^{\perp}$ and $F(D^{\perp})$ are Lagrangian distribution on (TM,Ω). In particular if N is a normal semi-invariant $\xi^{\perp}$-submanifold, then $T^{\perp}N$ is a Lagrangian submanifold in (TM,Ω).*

Proof. *We need to proof only (2). Let $X,Y \in D^{\perp}$, then $\Omega(X,Y) = g(FX,Y) = 0$ since $FX \in T^{\perp}N$ while $Y \in TN$. Again, let $X,Y \in D^{\perp}$, then $\Omega(X,Y) = g(FX,Y) = 0$ since $FX \in TN$ while $Y \in T^{\perp}N$* ■

4. INTEGRABILITY OF DISTRIBUTIONS ON SEMI-INVARIANT $\xi^{\perp}$- SUBMANIFOLD IN AFFINOR-METRIC GEOMETRY

Let N be a semi-invariant $\xi^{\perp}$-submanifold of a (g,F,μ)-manifold M. Then we use the Nijenhuis tensor field of F defined as follows [1]

$$N_F(X,Y) = [FX,FY] + F^2[X,Y] - F[FX,Y] - F[X,FY] \tag{4.1}$$

for any $X,Y \in TM$. In a similar way, the Nijenhuis tensor field of α on N is given by

$$N_\alpha(X,Y) = [X,\alpha Y] + \alpha^2[X,Y] - \alpha[\alpha X,Y] - \alpha[X,\alpha Y] \tag{4.2}$$

for any $X,Y \in TN$. We recall that a tensor field of type $(1,1)$ defines an integrable structure on a manifold if and only if its Nijenhuis tensor field vanishes identically on the manifold. Now we obtain necessary and sufficient conditions for the integrability of D and $D^{\perp}$ in terms of Nijenhuis tensor fields of F and α

Theorem 3 *Let N be a semi-invariant $\xi^{\perp}$-submanifold of a (g,F,μ)-manifold M. Then the following assertions are equivalent*

1. *D is and integrable distribution.*
2. *The Nijenhuis tensor field if α satisfies:*

$$Q \circ N_\alpha = 0, \quad \forall\ X,Y \in D. \tag{4.3}$$

3. *The Nijenhuis tensor fileds of F and α satisfy the equality, $N_F = N_\alpha$ on D*

Proof. *Firstly, we note that D is integrable if and only if*

$$Q([X,Y]) = 0, \quad \forall\ X,Y \in D. \tag{4.4}$$

Since the last three terms in the right side of (4.2) lie in D, so we deduce that:

$$Q \circ N_\alpha(X,Y) = Q([FX,FY]), \quad \forall X,Y \in D. \tag{4.5}$$

As M is nondegenerate we deduce that α is an automorphism on D. Thus the equivalence of (1) and (2) follows directly.

Next, we obtain for any $X,Y\in D$

$$N_F[X,Y]=N_\alpha(X,Y)+F\beta([X,Y])-\beta([\alpha X,Y])-\beta([X,\beta Y]). \tag{4.6}$$

If D is integrable, then the last three terms of (4.6) vanishes and this yields (3). Conversely, suppose that $N_F=N\alpha$ on D, then

$$F\beta[X,Y]=\beta([\alpha X,Y]+[X,\alpha Y]). \tag{4.7}$$

Obviously the right hand side of the previous equation in in $F(D^\perp)\subset T^\perp N$.On the other hand, the left hand side is in $F^2D^\perp$), we conclude that both sides in (4.7) must vanish.
Finaly, from $F^2Q([X,Y])=0$ and F^2 automorphism of TM we deduce (1). ■

Now , we consider $X,Y\in D^\perp$. Then taking into account that $\alpha X=\alpha Y=0$ we get

$$N_\alpha(X,Y)=F^2P[X,Y] \tag{4.8}$$

and this enables us to state the following theorem:

Theorem 4 *Let N be a semi-invariant $\xi^\perp$-submanifold of a (g,F,μ)-manifold M. Then $D^\perp$ is intetrable if and only if the Nijenhuis tensor filed of α vanishes identically on $D^\perp$.*

5. NATURAL FOLIATION ON A SEMI-INVARIANT $\xi^\perp$-SUBMANIFOLD IN AFFINOR-METRIC GEOMETRY

Let $\bar\nabla$ be the Levi-Civita connection on M with respect to the Riemannian metric g. Then F is a *parallel tensor field* on M if

$$\bar\nabla F=0. \tag{5.1}$$

In this section we study the geometry of semi-invariant $\xi^\perp$-submanifold of a (g,F,μ)-manifold with parallel tensor field F. First, we prove the following proposition

Proposition 5 *Let N be a semi-invariant $\xi^\perp$-submanifold of a (g,F,μ)-manifold M with parallel tenor field F. Then for all $X,Y\in D^\perp$*

$$A_{FX}Y-A_{FY}X=\alpha([X,Y]). \tag{5.2}$$

Proof. *By using the Weingarten formula and the parallelism condition, we get*

$$A_{FX}Y=\nabla^\perp{}_YFX-\nabla_YFX=\nabla^\perp_YFX-F(\bar\nabla_XY). \tag{5.3}$$

Writing a similar equation by interchanging X and Y and then using subtracting, we obtain

$$A_{FX}Y-A_{FY}X=\nabla^\perp{}_YFX-\nabla_YFX-\nabla^\perp_YFX+F([X,Y]), \tag{5.4}$$

since ∇ is a torsion free linear connection. Thus (5.2) is obtained by equalizing the tangent parts to N in the above equation.

Now, we can state the following main results:

Theorem 6 *Let N be a semi-invariant $\xi^\perp$-submanifold of a nondegenerate $(g,F\mu=+1)$-manifold with parallel tensor field F. Then the $F-\xi^\perp$-anti-invariant distribution $D^\perp$ is integrable.*
Proof. *For any $X,Y\in D^\perp$ and $Z\in D$ we have*

$$g(A_{FX}Y,Z)=-g(F\bar\nabla_YX,Z)=+\mu g(\bar\nabla_YX,FZ)=-\mu g(X,\bar\nabla_YFZ)= \tag{5.5}$$

$$=\mu^2g(FX,\bar\nabla_YZ)=\mu^2g(FX,[Y,Z]+\bar\nabla_ZY)=\mu^2g(FX,\bar\nabla_ZY).$$

Also, we have

$$g(A_{FY}X,Z)=\mu^2g(F\nabla_ZY,X)=\mu^3(\bar\nabla_ZY,FX). \tag{5.6}$$

Comparing (5.4) and (5.5) we deduce that for $\mu=+1$

$$g(A_{FX}Y-A_{FY}X,Z)=0 \tag{5.7}$$

$$A_{FX}Y-A_{FY}X\in D \tag{5.8}$$

and thus

$$A_{FX}Y-A_{FY}X=0. \tag{5.9}$$

which means that

$$A_{FX}Y-A_{FY}X\in D^\perp$$

On the other hand, form (5.2), we conclude that Finally, returning to (5.2) and taking into account that F is nondegenerate we deduce that

$$P[X,Y]=0,$$

that is, $D^{\perp}$ is integrable. ■

Regarding the integrability of D we prove the following:

Theorem 7 *Let N be a semi-invariant $\xi^{\perp}$-submanifold of a nondegenerate $(g, F\mu = +1)$-manifold with parallel tensor field F. Then the $F-\xi^{\perp}$-anti-invariant distribution D is integrable if and only if the second fundamental form h of N satisfies for any $X, Y \in D$ and $Z \in D^{\perp}$*

$$g(h(X, \alpha Y) - h(Y, \alpha X), FZ) = 0 \tag{5.10}$$

Proof. *By using the Gauss formula we deduce that*

$$\nabla_X \alpha Y + h(X, \alpha X) = \alpha(\nabla_X Y) + \beta(\nabla_X Y) + Fh(X,Y). \tag{5.11}$$

Now interchanging X and Y and then subtracting we obtain.

$$\nabla_X \alpha Y - \nabla_Y \alpha X + h(X, \alpha X) - h(Y, \alpha X) = \alpha(\nabla_X Y) + \beta(\nabla_X Y) + Fh(X,Y), \tag{5.12}$$

since h is symmetric and ∇ is a torsion free linear connection. Equalize the normal parts in the above equation, we obtain

$$h(X, \alpha X) - h(Y, \alpha X) = \beta([X,Y]). \tag{5.13}$$

Now, suppose that D is integrable; then (5.7) is immediately. Conversely, if (5.6) is satisfied, then with (5.10) we deduce that

$$-\mu g(Q[X,Y], F^2 Z) = 0. \tag{5.14}$$

Since F is nondegenerate we refer that F^2 is an automorphism of $D^{\perp}$ and hence D is integrable. ■

Now, for $\mu = +1$ we denote by $F^{\perp}$ the natural foliation defined by the F-anti-invariant distribution $D^{\perp}$ and call it the $F-$anti-invariant foliation on N. We recall that $F^{\perp}$ is called a totally geodesic foliation if each leaf of $F^{\perp}$ is totally geodesic immersed in N. Thus $F^{\perp}$ is totally geodesic if and only if the Levi-Civita connection ∇ of N satisfies for all $Y, Z \in D^{\perp}$

$$\nabla_Y Z \in (D^{\perp}). \tag{5.15}$$

Theorem 8 *Let N be a semi-invariant $\xi^{\perp}$-submanifold of a nondegenerate $(g, F\mu = +1)$-manifold M with parallel tensor field F. Then the following assertions are equivalent*

1. *The F-anti-invariant foliation is totally geodesic.*
2. *The second fundamental form h of N satisfies for all $X, Y \in D$ and $Y \in D^{\perp}$*

$$h(X,Y) \in \bar{D}. \tag{5.16}$$

3. *$D^{\perp}$ is A_V-invariant for any $V \in F(D^{\perp})$ that is we have for all $Y \in D^{\perp}$*

$$A_V Y \in D^{\perp}. \tag{5.17}$$

Proof. *We have for any $X \in D$ and $Y, Z \in D^{\perp}$*

$$g(\nabla_Y Z, FX) = g(\bar{\nabla}_Y Z, FX) = -\mu g(\bar{\nabla}_Y FZ, X) \tag{5.18}$$

$$= \mu g(A_{FZ} Y, X) = \mu g(h(X,Y), FZ).$$

Now, suppose $V \in F(D^{\perp})$ is totally geodesic, then the first term of (5.18) vanishes. Hence the last term in (5.14) vanishes which implies (3). Conversely, suppose (5.18) is satisfied. Then from (5.16) we deduce (5.11) since F is an automorphism of D. This proves the equivalence of (1) and (2). The equivalence of (2) and (3) is straightforward. ■

Finally, we can prove the following

Theorem 9 *Let N be a semi-invariant $\xi^{\perp}$-submanifold of a nondegenerate $(g, F\mu = +1)$-manifold M with parallel tensor field F. Then the F-invariant distribution D is integrable and the foliation $F(D^{\perp})$ defined by D is totally geodesic if and only if the second fundamental form h of N satisfies for all $X, Y \in D$*

$$h(X,Y) \in \bar{D} \tag{5.19}$$

Proof. *D is integrable and $F(D^{\perp})$ is totally geodesic if and only if for all $X, U \in D$*

$$\nabla_X U \in D. \tag{5.20}$$

This is equivalent to

$$g(\bar{\nabla}_X U, Z) = 0, \tag{5.21}$$

for all $Z \in D^{\perp}$. As F is an automorphism of D, we can write the above equality as follows

$$g(\bar{\nabla}_X FY, Z) = 0, \tag{5.22}$$

for all $X,Y \in D$ and $Z \in D^{\perp}$, which is equivalent to

$$g(\bar{\nabla}_X Y, Z) = 0. \tag{5.23}$$

By using the Gauss equation, the last realtion is equivalent to

$$g(h(X,Y), FZ) = 0 \tag{5.24}$$

which completes the proof of the theorem.

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