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Some Coupled Fixed Point Results in Discolated Quasi-b-Metric Spaces for Rational Type Contraction Mappings

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In this article, we have establish a coupled fixed point theorem satisfying rational contraction conditions in dislocated quasi-b-metric space. We illustrate our results with the help of an example. Our results extend and generalize very recent results of M. Ur Rahman and M. Sarwar [7].

Keywords: b- metric space, Discolated quasi-b-metric space, Rational contraction conditions, Fixed point, Coupled fixed point.

I. INTRODUCTION

In 1989, Bakhtin [3] introduced a very interesting concept b-metric space as an analog of a metric space.He proved the contraction mapping principle in b-metric space that generalized the renowned Banach contraction principle in metric spaces. Since then many mathematicians done several works on involving fixed point for single-valued and multi-valued operators in b-metric spaces(see for example[5, 6]). Sumati [8] has established the existence of a topology induced by a dislocated quasi metric and prove fixed point theorems for dislocated quasi metric spaces.
Recently, the authors of [1] has introduced dislocated quasi-b-metric space and derived related fixed point theorems by using cyclic contractions.
In 2006, Bhaskar and Lakshmikantham [10] introduced the notion of coupled fixed point and proved some fixed point theorem under certain condition, after the work of Bhaskar and Lakshmikantham [10] coupled fixed point theorems have been established by many authors in different type of spaces (see [2, 4, 11]).

In this paper, we have establish a coupled fixed point theorem satisfying rational contraction conditions in the context of dislocated quasi-b-metric space. Our main theorems extend and generalize many existing results in the literature. Moreover, we present several examples to illustrate the theorems.

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II. MATHEMATICAL PRELIMINARILY

Definition II.1 . *Let X be a nonempty set and $s \geq 1$ be a given real number. A functional $d : X \times X \longrightarrow \mathbb{R}^+$ is called a b-metric if for $x,y,z \in X$, the following conditions are satisfied:*

(1): $d(x,y) = 0$ *if and only if* $x = y$*,*

(3): $d(x,y) = d(y,x)$*,*

(4): $d(x,y) \leq s[d(x,u) + d(z,y)]$ *(b-triangular inequality).*

A pair (X,d) is called a b-metric space (with constant s).

Definition II.2 . *Let X be a nonempty set and $s \geq 1$ be a given real number. A functional $d : X \times X \longrightarrow \mathbb{R}^+$ is called quasi b-metric if for $x,y,z \in X$ the following conditions are satisfied:*

(1): $d(x,y) = d(y,x) = 0$ *if and only if* $x = y$,

(2): $d(x,y) \leq s[d(x,z) + d(z,y)]$.

A pair (X,d) *is called quasi-b-metric or shortly (qb-metric) space.*

Definition II.3 . *Let* X *be a nonempty set. A functional* $d : X \times X \longrightarrow \mathbb{R}^+$ *is called dislocated quasi-metric if for* $x,y,z \in X$ *the following conditions are satisfied:*

(1): $d(x,y) = d(y,x) = 0$ *implies that* $x = y$,

(2): $d(x,y) \leq d(x,z) + d(z,y)$.

A pair (X,d) *is called dislocated quasi-metric or shortly (dq-metric) space.*

Clearly every metric space is a dislocated metric space but the converse is not necessarily true (See [7]).

Definition II.4 . *Let* X *be a nonempty set and* $s \geq 1$ *be a given real number. A functional* $d : X \times X \longrightarrow \mathbb{R}^+$ *is called dislocated quasi b-metric if for* $x,y,z \in X$ *the following conditions are satisfied:*

(1): $d(x,y) = d(y,x) = 0$ *implies that* $x = y$,

(2): $d(x,y) \leq s[d(x,z) + d(z,y)]$.

A pair (X,d) *is called dislocated quasi-b-metric or shortly (dqb-metric) space.*

Note that every discolated quasi-metric space is a discolated quasi-b-metric space (with coefficient $s = 1$) and every quasi-b-metric space is a discolated quasi b-metric space. However the converse of the above implication is not necessarily true. The following gives some easy examples of dqb-metric.

Example 1[1]. Let $X = \mathbb{R}$ and let

$$d(x,y) = |x-y|^2 + \frac{|x|}{n} + \frac{|y|}{m},$$

where $n,m \in \mathbb{N} - \{1\}$ with $n \neq m$. Then (X,d) is a dislocated quasi-b-metric space with the coefficient $s = 2$, but since $d(1,1) \neq 0$, we have (X,d) is not a quasi-b-metric space, and we can prove that (X,d) is not a dislocated quasi-metric space[1].

Example 2[1]. Let $X = \mathbb{R}$ and let

$$d(x,y) = |x-y|^2 + 3|x| + 2|y|.$$

Then (X,d) is a dislocated quasi-b-metric space with the coefficient $s = 2$, but since $d(0,1) \neq d(1,0)$, we have (X,d) is not a b-metric space. It is obvious that (X,b) is not a dislocated quasi-metric space.

Definition II.5 . *Let* (X,d) *be a dq b-metric space and* (x_n) *be a sequence in* X*. Then we call*

(i): (x_n) *converges to* $x \in X$ *if*

$$\lim_{n \longrightarrow +\infty} d(x_n,x) = 0 = \lim_{n \longrightarrow +\infty} d(x,x_n)$$

In this case x *is called a dqb-limit of* (x_n)*, and we write it as* $x \longrightarrow x$ $(n \longrightarrow +\infty)$.

(ii): (x_n) *is a dqb-Cauchy sequence if*

$$\lim_{n,m \longrightarrow +\infty} d(x_n,x_m) = 0 = \lim_{n,m \longrightarrow +\infty} d(x_m,x_n)$$

(iii): (X,d) *is dqb-complete if every dqb-Cauchy sequence is convergent in* X.

Lemma II.1 . *Limit of convergent sequence in dislocated quasi b-metric space is unique.*

Definition II.6 *[9].* *Let* (X,d) *be a dqb-metric space with* $\varepsilon > 0$, $x_0 \in X$. *The set* $\delta(x_0,\varepsilon) = \{x/x \in X, \max\left(d(x_0,x), d(x,x_0)\right) < \varepsilon\}$ *is called dqb-open ball of radius* ε*,center* x_0 *and* $B_\varepsilon(x_0) = \{x_0\} \cup \delta(x_0,\varepsilon)$. *The set* $\overline{\delta}(x_0,\varepsilon) = \{x/x \in X, \max\left(d(x_0,x), d(x,x_0)\right) \leq \varepsilon\}$ *is called bdq-closed ball of radius* ε*,center* x_0 *and* $\overline{B}_\varepsilon(x_0) = \{x_0\} \cup \overline{\delta}(x_0,\varepsilon)$.

Definition II.7 *[10]. An element* $(x,y) \in X^2$ *is called coupled fixed point of the mapping* $T : X \times X \longrightarrow X$ *if* $T(x,y) = x$ *and* $T(y,x) = y$ *for* $x,y \in X$.

III. MAIN RESULT

Theorem III.1 . *Let* (X,d) *be a complete dislocated quasi b-metric space with coefficient* $s \geq 1$. $T : X \times X \longrightarrow X$ *be a* *continuous mapping satisfying the following rational contractive conditions*

$$\begin{array}{rl} & d(T(x,y),T(u,v)) \\ \leq & \alpha[d(x,u)+d(y,v)]+\frac{\beta}{s}\frac{d(x,T(x,y)).d(x,T(u,v))}{1+d(x,u)+d(y,v)} \\ & +\gamma\frac{d(x,T(x,y)).d(u,T(u,v))}{1+d(x,u)} \end{array} \quad (1)$$

for all $x,y,u,v \in X$ and α, β and γ are non-negative constants with $2\alpha s+\beta(s+1)+\gamma<1$. Then T has a unique coupled fixed point in $X \times X$.

Proof. Let x_0 and y_0 are arbitrary in X, we define the sequences (x_n) and (y_n) as following,

$$x_{n+1}=T(x_n,y_n) \ \ and \ \ y_{n+1}=T(y_n,x_n) \ \ for \ \ n \in \mathbb{N}$$

Consider

$$d(x_n,x_{n+1})=d(T(x_{n-1},y_{n-1}),T(x_n,y_n))$$

Now by (1) we have

$$\begin{array}{rcl} d(x_n,x_{n+1}) & \leq & \alpha[d(x_{n-1},x_n)+d(y_{n-1},y_n)] \\ & & +\frac{\beta}{s}\frac{d(x_{n-1},T(x_{n-1},y_{n-1})).d(x_{n-1},T(x_n,y_n))}{1+d(x_{n-1},x_n)+d(y_{n-1},y_n)} \\ & & +\gamma\frac{d(x_{n-1},T(x_{n-1},y_{n-1})).d(x_n,T(x_n,y_n))}{1+d(x_{n-1},x_n)} \end{array}$$

Using the definition of the sequences (x_n) and (y_n) we have

$$\begin{array}{rcl} d(x_n,x_{n+1}) & \leq & \alpha[d(x_{n-1},x_n)+d(y_{n-1},y_n)] \\ & & +\frac{\beta}{s}\frac{d(x_{n-1},x_n).d(x_{n-1},x_{n+1})}{1+d(x_{n-1},x_n)+d(y_{n-1},y_n)} \\ & & +\gamma\frac{d(x_{n-1},x_n).d(x_n,x_{n+1})}{1+d(x_{n-1},x_n)} \end{array}$$

Simplifying we have

$$\begin{array}{rcl} d(x_n,x_{n+1}) & \leq & \alpha[d(x_{n-1},x_n)+d(y_{n-1},y_n)] \\ & & +\frac{\beta}{s}d(x_{n-1},x_{n+1})+\gamma d(x_n,x_{n+1}) \\ & \leq & \alpha[d(x_{n-1},x_n)+d(y_{n-1},y_n)] \\ & & +\frac{\beta}{s}.s\big[d(x_{n-1},x_n)+d(x_n,x_{n+1})\big]+\gamma d(x_n,x_{n+1}) \end{array}$$

Simplification yeilds

$$d(x_n,x_{n+1}) \leq \frac{\alpha+\beta}{1-(\beta+\gamma)}d(x_{n-1},x_n)+\frac{\alpha}{1-(\beta+\gamma)}d(y_{n-1},y_n) \quad (2)$$

Similarly we can show that

$$d(y_n,y_{n+1}) \leq \frac{\alpha+\beta}{1-(\beta+\gamma)}d(y_{n-1},y_n)+\frac{\alpha}{1-(\beta+\gamma)}d(x_{n-1},x_n) \quad (3)$$

Adding (2) and (3) we have

$$[d(x_n,x_{n+1})+d(y_n,y_{n+1})] \leq \frac{2\alpha+\beta}{1-(\beta+\gamma)}[d(x_{n-1},x_n)+d(y_{n-1},y_n)$$

Since $2s\alpha+(s+1)\beta+\gamma<1$, so $sh<1$ with $h=\frac{2\alpha+\beta}{1-(\beta+\gamma)}<1$. Therefore the above inequality becomes,

$$[d(x_n,x_{n+1})+d(y_n,y_{n+1})] \leq h.[d(x_{n-1},x_n)+d(y_{n-1},y_n)$$

Also

$$[d(x_n,x_{n+1})+d(y_n,y_{n+1})] \leq h^2[d(x_{n-2},x_{n-1})+d(y_{n-2},y_{n-1})$$

Similarly proceeding we have

$$\delta_{n+1}=[d(x_n,x_{n+1})+d(y_n,y_{n+1})] \leq h^n[d(x_0,x_1)+d(y_0,y_1)=h^n\delta_1$$

Now by the b-triangular inequality we have:

$$\begin{array}{rl} & d(x_n,x_m)+d(y_n,y_m) \\ \leq & s\big(d(x_n,x_{n+1})+d(x_{n+1},x_m)+d(y_n,y_{n+1})+d(y_{n+1},y_m)\big) \\ \leq & s\big(d(x_n,x_{n+1})+d(y_n,y_{n+1})\big) \\ & +s^2\big(d(x_{n+1},x_{n+2})+d(y_{n+1},y_{n+2})\big) \\ & +s^2\big(d(x_{n+2},x_m)+d(y_{n+2},y_m)\big) \\ \vdots & \\ \leq & s\delta_{n+1}+s^2\delta_{n+2}+....+s^{m-n}\delta_m \\ \leq & \Big(sh^n+s^2h^{n+1}+...+s^{m-n}h^{m-1}\Big)\delta_1 \\ = & sh^n\big(1+sh+...+(sh)^{m-n-1}\big)\delta_1 \\ \leq & sh^n\frac{1}{1-sh}\delta_1 \end{array}$$

Since $h<1$, then

$$\lim_{n,m\longrightarrow+\infty}[d(x_n,x_m)+d(y_n,y_m)]=0$$

which implies $\lim_{n,m\longrightarrow+\infty}d(x_n,x_m)=0$ and $\lim_{n,m\longrightarrow+\infty}d(y_n,y_m)=0$ Similarly proceeding we have $\lim_{n,m\longrightarrow+\infty}d(x_m,x_n)=0$ and $\lim_{n,m\longrightarrow+\infty}d(y_m,y_n)=0$ Thus (x_n) and (y_n) are Cauchy sequences in complete dislocated quasi-metric space X. So there must exist $w,z \in X$ such that $\lim_{n\longrightarrow+\infty}x_n=w$ and $\lim_{n\longrightarrow+\infty}y_n=z$. Also since T is continuous and $T(x_n,y_n)=x_{n+1}$ so taking limit $n \longrightarrow +\infty$. We have

$$\lim_{n\longrightarrow+\infty}T(x_n,y_n)=\lim_{n\longrightarrow+\infty}x_{n+1}$$

$$T(\lim_{n\longrightarrow+\infty}x_n,\lim_{n\longrightarrow+\infty}y_n)=\lim_{n\longrightarrow+\infty}x_{n+1}$$

$$T(w,z)=w$$

Also from $T(y_n,x_n)=y_{n+1}$. We can show that $d(z,w)=z$. Thus $(w,z)\in X\times X$ is the coupled fixed point of T in X.

Uniqueness. Let (w,z) and (w_1,z_1) are two distinct coupled fixed points of T in $X\times X$. Then by use of (1) we have

$$\begin{aligned}
& d(w,w)=d(T(w,z),T(w,z))\\
\leq\ & \alpha[d(w,w)+d(z,z)]+\frac{\beta}{s}\frac{d(w,T(w,z)).d(w,T(w,z))}{1+d(w,w)+d(z,z)}\\
& +\gamma\frac{d(w,T(w,z)).d(w,T(w,z))}{1+d(w,w)}\\
\leq\ & \alpha[d(w,w)+d(z,z)]+\frac{\beta}{s}\frac{d(w,w).d(w,w)}{1+d(w,w)+d(z,z)}\\
& +\gamma\frac{d(w,w)).d(w,w))}{1+d(w,w)}\\
\leq\ & \alpha[d(w,w)+d(z,z)]+\frac{\beta}{s}d(w,w)+\gamma d(w,w))\\
\leq\ & \alpha[d(w,w)+d(z,z)]+\beta[d(w,w)+d(z,z)]\\
& +\gamma d(w,w)
\end{aligned}$$

Then simplifying we have

$$d(w,w)\leq(\alpha+\beta)[d(w,w)+d(z,z)]+\gamma d(z,z) \tag{4}$$

Similarly we can show that

$$d(z,z)\leq(\alpha+\beta)[d(w,w)+d(z,z)]+\gamma d(w,w) \tag{5}$$

Adding (4) and (5) we have

$$[d(w,w)+d(z,z)]\leq(2\alpha+2\beta+\gamma)[d(w,w)+d(z,z)]$$

Since $2\alpha+2\beta+\gamma<1$ so the above inequality is possible only if $[d(w,w)+d(z,z)]=0$ Implies

$$d(w,w)=d(z,z)=0 \tag{6}$$

Now consider

$$\begin{aligned}
d(w,w_1)\ &=\ d(T(w,z),T(w_1,z_1))\\
&\leq\ \alpha[d(w,w_1)+d(z,z_1)]\\
&+\ \frac{\beta}{s}\frac{d(w,T(w,z)).d(w,T(w_1,z_1))}{1+d(w,w_1)+d(z,z_1)}\\
&+\ \gamma\frac{d(w,T(w,z)).d(w_1,T(w_1,z_1))}{1+d(w,w_1)}\\
&\leq\ \alpha[d(w,w_1)+d(z,z_1)]\\
&+\ \frac{\beta}{s}\frac{d(w,w).d(w,w_1)}{1+d(w,w_1)+d(z,z_1)}\\
&+\ \gamma\frac{d(w,w).d(w_1,w_1))}{1+d(w,w_1)}
\end{aligned}$$

Now using (6) we have the following

$$d(w,w_1)\leq\alpha[d(w,w_1)+d(z,z_1)] \tag{7}$$

By following similar procedure we can get

$$d(z,z_1)\leq\alpha[d(z,z_1)+d(w,w_1)] \tag{8}$$

Adding (7) and (8) we have

$$d(w,w_1)+d(z,z_1)\leq 2\alpha[d(w,w_1)+d(z,z_1)]$$

Since $2\alpha<1$ so the above inequality is possible only if

$$d(w,w_1)+d(z,z_1)=0$$

Which implies that

$$d(w,w_1)=d(z,z_1)=0$$

Similarly we can show that

$$d(w_1,w)=d(z_1,z)=0$$

Implies $w=w_1$ and $z=z_1$ Hence

$$(w,z)=(w_1,z_1)$$

Thus coupled fixed point of T in $X\times X$ is unique.

Remark. Theorem III.1 is a generalization of Theorem 2.1 (taking $s=1$) of M. Ur Rahman and M. Sarwar [7].

We deduce the following corollaries from Theorem III.1.

Corollary III.1 . *Let (X,d) be a complete dislocated quasi b-metric space with coefficient $s\geq 1$. $T:X\times X\longrightarrow X$ be a continuous mapping satisfying the following rational contractive conditions*

$$d(T(x,y),T(u,v))\leq\alpha[d(x,u)+d(y,v)]+\frac{\beta}{s}\frac{d(x,T(x,y)).d(x,T(u,v))}{1+d(x,u)+d(y,v)} \tag{9}$$

for all $x,y,u,v\in X$ and α, β are non-negative constants with $2\alpha s+\beta(s+1)<1$. Then T has a unique coupled fixed point in $X\times X$.

Corollary III.2 . *Let (X,d) be a complete dislocated quasi b-metric space with coefficient $s\geq 1$. $T:X\times X\longrightarrow X$ be a continuous mapping satisfying the following rational contractive conditions*

$$d(T(x,y),T(u,v))\leq\alpha[d(x,u)+d(y,v)]+\frac{\beta}{s}\frac{d(x,T(x,y)).d(x,T(u,v))}{1+d(x,u)} \tag{10}$$

for all $x,y,u,v\in X$ and α, β are non-negative constants with $2\alpha s+\beta(s+1)<1$. Then T has a unique coupled fixed point in $X\times X$.

Corollary III.3 . *Let (X,d) be a complete dislocated quasi b-metric space with coefficient $s \geq 1$. $T : X \times X \longrightarrow X$ be a continuous mapping satisfying the following rational contractive conditions*

$$d(T(x,y),T(u,v)) \leq \alpha[d(x,u)+d(y,v)] \quad (11)$$

for all $x,y,u,v \in X$ and $\alpha > 0$ with $2\alpha s < 1$. Then T has a unique coupled fixed point in $X \times X$.

Example. Let $X = [-1,1]$. Define $d : X \times X \longrightarrow \mathbb{R}_+$ by

$$d(x,y) = \mid x-y \mid^2 + 3 \mid x \mid + 2 \mid y \mid .$$

for all $x,y \in X$. Then (X,d) is a complete dislocated quasi b-metric space with the coefficient $s = 2$. Owing to $d(1,1) \neq 0$, hence (X,d) is not a quasi-b-metric space. Also, it is not a dislocated b-metric space because of $d(1,0) \neq d(0,1)$.

Define a continuous self-map $T : X \times X \longrightarrow X$ by $T(x,y) = \frac{1}{6}xy$ for all $x,y \in X$. Since
$\mid xy-uv \mid \leq \mid x-u \mid . \mid y-v \mid, \mid xy \mid \leq \mid x \mid + \mid y \mid$ and $2xy \leq x^2+y^2$
Hold for all $x,y,u,v \in X$. Then

$$\begin{aligned} d(T(x,y),T(u,v)) &= d(\tfrac{1}{6}xy, \tfrac{1}{6}uv) \\ &= \tfrac{1}{36} \mid xy-uv \mid^2 + \tfrac{3}{6} \mid xy \mid + \tfrac{2}{6} \mid uv \mid \\ &\leq \tfrac{1}{36}\big(2 \mid x-u \mid^2 + 2 \mid y-v \mid^2\big) \\ &\quad + \tfrac{3}{6}(\mid x \mid + \mid y \mid) + \tfrac{2}{6}(\mid u \mid + \mid v \mid) \\ &\leq \tfrac{1}{6}\big(\mid x-u \mid^2 + 3 \mid x \mid + 2 \mid u \mid \big) \\ &\quad + \tfrac{1}{6}\big(\mid y-v \mid^2 + 3 \mid y \mid + 2 \mid v \mid \big) \\ &= \tfrac{1}{6}[d(x,u)+d(y,v)] \end{aligned}$$

So for $\alpha = \dfrac{1}{6}$ and $\beta = \gamma = 0$ all the conditions of Theorem III.1 are satisfied having $(0,0) \in X \times X$ is the unique coupled fixed point of T in $X \times X$.

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