Asian Journal of Mathematics and Physics
Volume 2, Issue 1, 2018, 14-21

A New Generalization for Jacobsthal and Jacobsthal Lucas Sequences

S. Uygun[1]

[1] *Department of Mathematics, Science and Art Faculty, Gaziantep University, Campus, Gaziantep, Turkey*[*]

(Received 12 Oct. 2017)

In this study, we define *new generalizations for* Jacobsthal and Jacobsthal Lucas *sequences called* $p(x)-$Jacobsthal and $p(x)-$Jacobsthal *Lucas polynomial sequences where* $p(x)$ *is any real valued polynomial. We get the Binet formulae, generating functions, and some important properties for these sequences. And then we describe a matrix whose elements are of* $p(x)-$Jacobsthal *terms. By using this matrix we derive some properties for* $p(x)-$Jacobsthal and $p(x)-$Jacobsthal *Lucas polynomial sequences*

Keywords: Jacobsthal sequence, Jacobsthal *Lucas* sequence, Binet formula, Generating functions.

I. INTRODUCTION AND PRELIMINARIES

Special integer sequences are encountered in different branches of science, art, nature, the structure of our body. So it is a popular topic in applied mathematics. Generalization of special integer sequences has been studied by many researchers by using various approaches, such as changing the initial conditions or adding new parameters to the recurrence relations, changing the recurrence relation with respect to parity of index n. There are so many studies about Fibonacci numbers (the first known special integer sequence). A generalization of Fibonacci numbers, the Fibonacci polynomials are studied by Catalan and defined by the recurrence relation $F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$, $F_0(x) = 0$, $F_1(x) = 1$. Similarly another generalization sequence, the Lucas polynomials are defined by $l_n(x) = xl_{n-1}(x)+l_{n-2}(x)$, $l_0(x) = 2$, $l_1(x) = x$. Swammy in [10] defined the generalized Fibonacci and Lucas polynomials and their diagonal polynomials. In [11] Catalini gave some properties of bivariate Fibonacci and bivariate Lucas polynomials. Djordjevic considered the generating functions, explicit formulas for generalized Fibonacci and Lucas polynomials in [12].

Our paper is about Jacobsthal and Jacobsthal Lucas numbers. So first of all we want to give the recurrence relations for Jacobsthal and Jacobsthal Lucas sequences as $j_n = j_{n-1} + 2j_{n-2}$, $j_0 = 0$, $j_1 = 1$ and $c_n = c_{n-1} + 2c_{n-2}$, $c_0 = 2$, $c_1 = 1$ for $n \geq 2$, respectively in [3]. There are some generalizations of these integer sequences. For example,; the Jacobsthal and the Jacobsthal Lucas polynomial sequences sequences are defined recurrently by $\hat{j}_n(x) = \hat{j}_{n-1}(x) + 2x\hat{j}_{n-2}(x)$, $\hat{j}_0(x) = 0$, $\hat{j}_1(x) = 1$, $n \geq 2$ and $\hat{c}_n(x) = \hat{c}_{n-1}(x) + 2x\hat{c}_{n-2}(x)$, $\hat{c}_0 = 2$, $\hat{c}_1 = 1$, $n \geq 2$ in [5]. And another generalization of Jacosthal sequences is given in [7] as $j_n(s,t) = sj_{n-1}(s,t) + 2tj_{n-2}(s,t)$, $j_0(s,t) = 0$, $j_1(s,t) = 1$ and $c_n(s,t) = sc_{n-1}(s,t) + 2tc_{n-2}(s,t)$, $c_0(s,t) = 2$, $c_1(s,t) = 1$ for $n \geq 2$.

The object of this paper is to define a new generalization of Jacobsthal and Jacobsthal Lucas sequences by

[*] suygun@gantep.edu.tr

using polynomials as called $p(x)$-Jacobsthal polynomial $J_{p,n}(x)$ and $p(x)$-Jacobsthal Lucas polynomial $C_{p,n}(x)$ and record some basic properties of $J_{p,n}(x)$ and $C_{p,n}(x)$.

II. THE P(X)-JACOBSTHAL AND P(X)-JACOBSTHAL LUCAS POLYNOMIALS AND THEIR PROPERTIES

Definition 1 *Assume that $p(x)$ is a polynomial with real coefficients and $n \geq 2$ any integer. The $p(x)$-Jacobsthal polynomial $\left\{J_{p,n}(x)\right\}_{n\in\mathbb{N}}$ sequences are described by using the following recurrence relation*

$$J_{p,n}(x) = p(x)J_{p,n-1}(x) + 2J_{p,n-2}(x) \tag{2.1}$$

with initial conditions are $J_{p,0}(x) = 0$, $J_{p,1}(x) = 1$, and $p(x)$-Jacobsthal Lucas polynomial $\left\{C_{p,n}(x)\right\}_{n\in\mathbb{N}}$ sequences

$$C_{p,n}(x) = p(x)C_{p,n-1}(x) + 2C_{p,n-2}(x) \tag{2.2}$$

with initial conditions are $C_{p,0}(x) = 2$, $C_{p,1}(x) = p(x)$.

Initially, the polynomials are defined for only positive terms but their existence for $n < 0$ is readily obtained, yielding

$$J_{p,-n}(x) = -J_{p,n}(x)/(-2)^n,$$
$$C_{p,-n}(x) = C_{p,n}(x)/(-2)^n.$$

The first some terms of $p(x)$-Jacobsthal polynomials are $J_{p,1}(x) = 1$, $J_{p,2}(x) = p(x)$, $J_{p,3}(x) = p^2(x) + 2$, $J_{p,4}(x) = p^3(x) + 4p(x)$.

And the first some terms of $p(x)$-Jacobsthal Lucas polynomials are $C_{p,1}(x) = p(x)$, $C_{p,2}(x) = p^2(x) + 4$, $C_{p,3}(x) = p^3(x) + 6p(x)$, $C_{p,4}(x) = p^4(x) + 8p^2(x) + 8$.

Special numerical choices for $p(x)$-Jacobsthal polynomial and $p(x)$-Jacobsthal Lucas polynomial are: If $p(x) = 1$, then we get classic Jacobsthal and Jacobsthal Lucas sequences. If $p(x) = k$, then we get classic $k-$Jacobsthal and $k-$Jacobsthal Lucas sequences.The characteristic equation of recurrence relation for $p(x)$-Jacobsthal polynomial and $p(x)$-Jacobsthal Lucas polynomial is

$$r^2 - p(x)r - 2 = 0.$$

The roots of the characteristic equation are

$$\alpha(x) = \frac{p(x) + \sqrt{p^2(x) + 8}}{2}, \quad \beta(x) = \frac{p(x) - \sqrt{p^2(x) + 8}}{2} \tag{2.3}$$

with the following properties

$$\alpha(x)+\beta(x) = p(x), \quad \alpha(x)-\beta(x) = \sqrt{p^2(x) + 8}, \quad \alpha(x).\beta(x) = -2. \tag{2.4}$$

Lemma 2 *The Binet formulas for these sequences are*

$$J_{p,n}(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)}, \tag{2.5}$$
$$C_{p,n}(x) = \alpha^n(x) + \beta^n(x). \tag{2.6}$$

Proof. The proof is obtained easily by using the values of the first two terms of $J_{p,n}(x)$ and $C_{p,n}(x)$. ■

Theorem 3 ***(The generating function)*** *Let i, j any natural numbers and $\left|\alpha^i(x)t\right| < 1$ and $\left|\beta^i(x)t\right| < 1$. Then the generating functions of these sequences for different indices are obtained as*

$$\sum_{n=0}^{\infty} J_{p,in+j}(x)t^n = \frac{J_{p,j}(x) + (-2)^j J_{p,i-j}(x)t}{1 - C_{p,i}(x)t + (-2)^i t^2}, \tag{2.7}$$
$$\sum_{n=0}^{\infty} C_{p,in+j}(x)t^n = \frac{C_{p,j}(x) - (-2)^j C_{p,i-j}(x)t}{1 - C_{p,i}(x)t + (-2)^i t^2}. \tag{2.8}$$

Proof. By using Binet formula for $p(x)$-*Jacobsthal* polynomial *sequence, we get*

$$\begin{aligned}
\sum_{n=0}^{\infty} J_{p,in+j}(x)t^n &= \sum_{n=0}^{\infty} \frac{\alpha^{in+j}(x) - \beta^{in+j}(x)}{\alpha(x) - \beta(x)} t^n \\
&= \frac{1}{\alpha(x) - \beta(x)} \left[\alpha^j \sum_{n=0}^{\infty} \left(\alpha^i t\right)^n - \beta^j \sum_{n=0}^{\infty} \left(\beta^i t\right)^n\right] \\
&= \frac{1}{\alpha(x) - \beta(x)} \left[\frac{\alpha^j}{1 - \alpha^i(x)t} - \frac{\beta^j}{1 - \beta^i(x)t}\right] \\
&= \frac{\left(\alpha^j(x) - \beta^j(x)\right) + (-2)^j \left(\alpha^{i-j}(x) - \beta^{i-j}(x)\right) t}{\alpha(x) - \beta(x) \left(1 - t \left(\alpha^i(x) + \beta^i(x)\right) + t^2 (-2)^i\right)} \\
&= \frac{J_{p,j}(x) + (-2)^j J_{p,i-j}(x)t}{1 - C_{p,i}(x)t + (-2)^i t^2}.
\end{aligned}$$

The other part of the proof is done by using the same method. ■

Some examples *for different values of* i, j are given as

$$\sum_{n=0}^{\infty} J_{p,n}(x)t^n = \frac{t}{1-p(x)t-2t^2},$$
$$\sum_{n=0}^{\infty} C_{p,n}(x)t^n = \frac{2-p(x)t}{1-p(x)t-2t^2}$$
$$\sum_{n=0}^{\infty} J_{p,2n}(x)t^n = \frac{p(x)t}{1-(p^2(x)+4)t+4t^2},$$
$$\sum_{n=0}^{\infty} C_{p,2n}(x)t^n = = \frac{2-(p^2(x)+4)t}{1-(p^2(x)+4)t+4t^2}$$
$$\sum_{n=0}^{\infty} J_{p,2n+1}(x)t^n = \frac{1-2t}{1-(p^2(x)+4)t+4t^2},$$
$$\sum_{n=0}^{\infty} C_{p,2n+1}(x)t^n = \frac{p(x)+2p(x)t}{1-(p^2(x)+4)t+4t^2}$$

Theorem 4 ***(Explicit closed form)*** *Let* $n \geq 1$

$$J_{p,n}(x) = 2^{-n+1}\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor}\binom{n}{2i+1}(p(x))^{n-2i-1}\left(\sqrt{p^2(x)+8}\right)^{2i} \quad (2.9)$$
$$C_{p,n}(x) = 2\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n}{2i}(p(x))^{n-2i}\left(p^2(x)+8\right)^{i} \quad (2.10)$$

Proof.

$$\begin{aligned}\alpha^n(x)-\beta^n(x) &= [(p(x)+\sqrt{p^2(x)+8})^n-(p(x)-\sqrt{p^2(x)+8})^n]/2^n\\ &= 2^{-n}\sum_{i=0}^{n}\binom{n}{i}(p(x))^{n-i}\begin{bmatrix}\left(\sqrt{p^2(x)+8}\right)^i\\ -\left(-\sqrt{p^2(x)+8}\right)^i\end{bmatrix}\\ &= 2^{-n+1}\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor}\binom{n}{2i+1}(p(x))^{n-2i-1}\left(\sqrt{p^2(x)+8}\right)^{2i+1}\end{aligned}$$

$$\begin{aligned}J_{p,n}(x) &= \frac{\alpha^n(x)-\beta^n(x)}{\alpha(x)-\beta(x)}\\ &= 2^{-n+1}\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor}\binom{n}{2i+1}(p(x))^{n-2i-1}\left(\sqrt{p^2(x)+8}\right)^{2i}\end{aligned}$$

$$\begin{aligned}\alpha^n(x)+\beta^n(x) &= [(p(x)+\sqrt{p^2(x)+8})^n+(p(x)-\sqrt{p^2(x)+8})^n]\\ &= \sum_{i=0}^{n}\binom{n}{i}(p(x))^{n-i}\begin{bmatrix}\left(\sqrt{p^2(x)+8}\right)^i\\ +\left(-\sqrt{p^2(x)+8}\right)^i\end{bmatrix}\\ &= 2\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n}{2i}(p(x))^{n-2i}\left(p^2(x)+8\right)^i\end{aligned}$$

■

Theorem 5 *We can also find explicit closed form by using the generating function for* $J_{p,n}(x)$ *and* $C_{p,n}(x)$. *Let* $n \geq 1$ *any integer,*

$$J_{p,n}(x) = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor}\binom{n-1-i}{i}(p(x))^{n-1-2i}2^i \quad (2.11)$$
$$C_{p,n}(x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor}\frac{n}{n-i}\binom{n-i}{i}(p(x))^{n-2i}2^i \quad (2.12)$$

Proof.

$$\begin{aligned}\sum_{n=0}^{\infty}J_{p,n}(x)t^n &= \frac{t}{1-p(x)t-2t^2} = t\sum_{n=0}^{\infty}\left(p(x)t+2t^2\right)^n\\ &= t\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}(p(x)t)^{n-i}\left(2t^2\right)^i\\ &= \sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}(p(x))^{n-i}\,2^i t^{n+i+1}\\ &= \sum_{n=0}^{\infty}\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor}\binom{n-i-1}{i}(p(x))^{n-2i-1}\,2^i t^n\end{aligned}$$

From the equality of both sides, the desired result obtained. ■

Corollary 6

$$\frac{dC_{p,n}(x)}{dx} = nJ_{p,n}(x)\frac{dp(x)}{dx}$$

Proof. From (2.12), we get

$$\begin{aligned}\frac{dC_{p,n}(x)}{dx} &= \left(\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor}\frac{n}{n-i}\binom{n-i}{i}(p(x))^{n-2i}2^i\right)'\\ &= \frac{dp(x)}{dx}\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor}\frac{n(n-2i)}{n-i}\binom{n-i}{i}(p(x))^{n-2i-1}2^i\\ &= \frac{dp(x)}{dx}\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor}\frac{n(n-2i)(n-i)!}{(n-i)(n-2i)!i!}(p(x))^{n-2i-1}2^i\\ &= \frac{dp(x)}{dx}\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor}\frac{n(n-i-1)!}{(n-2i-1)!i!}(p(x))^{n-2i-1}2^i\\ &= \frac{dp(x)}{dx}\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor}\binom{n-1-i}{i}n(p(x))^{n-1-2i}2^i\end{aligned}$$

■

(Important Relationships)

Important elementary relationships involving $J_{p,n}(x)$ and $C_{p,n}(x)$ follow with the aid of (2.1)-(2.6)

a) $J_{p,n}(x)C_{p,n}(x) = J_{p,2n}(x)$,

b) $C_{p,n}(x) = J_{p,n+1}(x) + 2J_{p,n-1}(x) = p(x)J_{p,n}(x) + 4J_{p,n-1}(x)$,

c) $(p^2(x)+8)J_{p,n}(x) = C_{p,n+1}(x) + 2C_{p,n-1}(x)$,

d) $p(x)J_{p,n}(x) + C_{p,n}(x) = 2J_{p,n+1}(x)$,

e) $(p^2(x)+8)J_{p,n}(x) + p(x)C_{p,n}(x) = 2C_{p,n+1}(x)$,

f) $\sqrt{p^2(x)+8}J_{p,n}(x) + C_{p,n}(x) = 2\alpha^n$,

g) $\sqrt{p^2(x)+8}J_{p,n}(x) - C_{p,n}(x) = -2\beta^n$,

h) $C^2_{p,n+2}(x) + 2C^2_{p,n+1}(x) = C_{p,2n+4}(x) + 2C_{p,2n+2}(x)$,

i) $J^2_{p,n+1}(x) + 2J^2_{p,n}(x) = J_{p,2n+1}(x)$.

j) $C_{p,2n}(x) = J^2_{p,n}(x)\left(p^2(x)+8\right) + 2(-2)^n$

k) $C^2_{p,n}(x) = C_{p,2n}(x) + 2(-2)^n$

l) $\left(p^2(x)+8\right)J^2_{p,n}(x) = C_{p,2n}(x) - 2(-2)^n$

m) $C_{p,3n}(x) = C_{p,n}(x)(C_{p,2n}(x) - (-2)^n)$

n) $J_{p,3n}(x) = J_{p,n}(x)(C_{p,2n}(x) + (-2)^n)$

o) $C^2_{p,n}(x) - (p^2(x)+8)J^2_{p,n}(x) = 4(-2)^n$

p) $J^2_{p,n+1}(x) - 4J^2_{p,n-1}(x) = p(x)J_{p,2n}(x)$.

Theorem 7 ***(D'ocagne's property)***

Let $n \geq m$ and $n, m \in \mathbb{Z}^+$. For $p(x)$- *Jacobsthal* and $p(x)$- *Jacobsthal Lucas* polynomial *sequences*, we have

$$J_{p,m+1}(x)J_{p,n}(x) - J_{p,m}(x)J_{p,n+1}(x) = (-2)^m J_{p,n-m}(x).$$

$$C_{p,m+1}(x)C_{p,n}(x) - C_{p,m}(x)C_{p,n+1}(x) = -\sqrt{p^2(x)+8}(-2)^m C_{p,n-m}(x).$$

Proof. By using Binet formula, we have

$$\begin{aligned}
&J_{p,m+1}(x)J_{p,n}(x) - J_{p,m}(x)J_{p,n+1}(x)\\
&= \frac{\alpha^{m+1}(x)-\beta^{m+1}(x)}{\alpha(x)-\beta(x)}\frac{\alpha^{n}(x)-\beta^{n}(x)}{\alpha(x)-\beta(x)}\\
&\quad -\frac{\alpha^{m}(x)-\beta^{m}(x)}{\alpha(x)-\beta(x)}\frac{\alpha^{n+1}(x)-\beta^{n+1}(x)}{\alpha(x)-\beta(x)}\\
&= \frac{1}{(\alpha(x)-\beta(x))^2}\begin{bmatrix} \alpha^n(x)\beta^m(x)(\alpha(x)-\beta(x)) \\ -\alpha^m(x)\beta^n(x)(\alpha(x)-\beta(x)) \end{bmatrix}\\
&= \frac{1}{\alpha(x)-\beta(x)}\left[(\alpha(x)\beta(x))^m(\alpha(x)^{n-m}-\beta(x)^{n-m})\right]\\
&= \frac{1}{\alpha(x)-\beta(x)}\left[(-2)^m\left(\alpha(x)^{n-m}-\beta(x)^{n-m}\right)\right].
\end{aligned}$$

The D'ocagne's property for $p(x)$- *Jacobsthal Lucas* polynomial *sequences can readily seen by using the same method.* ■

Theorem 8 ***(Catalan's property)***

Assume that $n, r \in \mathbb{Z}^+$. For $p(x)$- *Jacobsthal* and $p(x)$- *Jacobsthal Lucas* polynomial *sequences*, we have

$$J_{p,n+r}(x)J_{p,n-r}(x) - J^2_{p,n}(x) = -(-2)^{n-r}J^2_{p,r}(x)$$

$$C_{p,n+r}(x)C_{p,n-r}(x) - C^2_{p,n}(x) = (-2)^{n-r}J^2_{p,r}(x)(p^2(x)+8).$$

Proof. The proof is readily obtained by Binet formula. ■

Theorem 9 ***(Cassini's property or Simpson property)***

For $n \in \mathbb{Z}^+$, we have

$$J_{p,n+1}(x)J_{p,n-1}(x) - J^2_{p,n}(x) = -(-2)^{n-1}$$

$$C_{p,n+1}(x)C_{p,n-1}(x) - C^2_{p,n}(x) = (-2)^{n-1}(p^2(x)+8).$$

We get these properties by substituting 1 for r in Catalan property.

Theorem 10

$$C_{p,4n}(x) + 2^{2n+1} = (p^2(x)+8)J^2_{p,n}(x)$$

$$2C_{p,2n}(x) = (p^2(x)+8)J_{p,n+1}(x)J_{p,n-1}(x) - C_{p,n+1}(x)C_{p,n-1}(x)$$

Proof. The proof is readily obtained by Binet formula. ■

Theorem 11 *By this theorem we get new relations between the roots* α, β *and* $p(x)$*- Jacobsthal and* $p(x)$*-Jacobsthal Lucas polynomial sequences.*

$$\alpha^n(x) = \alpha(x)J_{p,n}(x) + 2J_{p,n-1}(x),$$

$$\beta^n(x) = \beta(x)J_{p,n}(x) + 2J_{p,n-1}(x),$$

$$\sqrt{p^2(x)+8}\alpha^n(x) = \alpha(x)C_{p,n}(x) + 2C_{p,n-1}(x),$$

$$-\sqrt{p^2(x)+8}\beta^n(x) = \beta(x)C_{p,n}(x) + 2C_{p,n-1}(x).$$

Proof. We prove by using Binet formula and the product of the roots,

$$\begin{aligned} &\beta(x)J_{p,n}(x) + 2J_{p,n-1}(x) \\ &= \beta(x)\frac{\alpha^n(x)-\beta^n(x)}{\alpha(x)-\beta(x)} + 2\frac{\alpha^{n-1}(x)-\beta^{n-1}(x)}{\alpha(x)-\beta(x)} \\ &= \frac{1}{\alpha(x)-\beta(x)}\left[\beta(x)\left(\alpha^n(x)-\beta^n(x)\right) + 2\left(\alpha^{n-1}(x)-\beta^{n-1}(x)\right)\right] \\ &= \frac{1}{\alpha(x)-\beta(x)}\left(-2\alpha^{n-1}(x) - \beta^{n+1}(x) + 2\alpha^{n-1}(x) - 2\beta^{n-1}(x)\right) \\ &= \frac{1}{\alpha(x)-\beta(x)}\left[-\beta^{n-1}(x)\left(\beta^2(x)+2\right)\right] = \beta^n(x). \end{aligned}$$

Other proofs can be done by using the same way. ■

Theorem 12 *For* $p(x)$*- Jacobsthal polynomial sequences, we get*

$$J_{p,4n+k}(x) - 2^{2n}.J_{p,k}(x) = J_{p,2n}(x)C_{p,2n+k}(x),$$

$$J_{p,4n+k}(x) + 2^{2n}.J_{p,k}(x) = C_{p,2n}(x)J_{p,2n+k}(x),$$

$$J_{p,3n+k}(x) - (-2)^n\,.J_{p,n+k}(x) = J_{p,n}(x)C_{p,2n+k}(x),$$

$$J_{p,3n+k}(x) + (-2)^n\,.J_{p,n+k}(x) = C_{p,n}(x)J_{p,2n+k}(x),$$

where $n \geq 1$, $p \geq 0$.

It can be proved by using Binet formulas as the following theorem.

Theorem 13 *For* $p(x)$*-Jacobsthal Lucas polynomial sequence, we have*

$$C_{p,4n+k}(x) - 2^{2n}.C_{p,k}(x) = (p^2(x)+8)J_{p,2n}(x)J_{p,2n+k}(x),$$

$$C_{p,4n+k}(x) + 2^{2n}.C_{p,n}(x) = C_{p,2n}(x)C_{p,2n+k}(x),$$

$$C_{p,3n+k}(x) - (-2)^n\,.C_{p,n+k}(x) = (p^2(x)+8)J_{p,n}(x)J_{p,2n+k}(x),$$

$$C_{p,3n+k}(x) + (-2)^n\,C_{p,n+k}(x) = .C_{p,n}(x).C_{p,2n+k}(x),$$

Theorem 14 *Assume that* $A_0(x) = [0]$ *and* $A_n(x)$ *is a nxn tridigional matrix defined as*

$$A_n(x) = \begin{bmatrix} 1 & i & & & & \\ 0 & p(x) & i & . & & \\ & i & p(x) & . & & \\ & & . & . & . & \\ & & & . & . & . \\ & & & & . & . & i \\ & & & & & i & p(x) \end{bmatrix}$$

where $i = \sqrt{-1}$ and $n \geq 0$. Then

$$\det A_n(x) = J_{p,n}(x).$$

Proof. The proof is made by mathematical induction method. For $n = 0, 1$, we have $\det A_0(x) = J_{p,0}(x) = 0$ and $\det A_1(x) = J_{p,1}(x) = 1$. Assume that $\det A_{n-1}(x) = J_{p,n-1}(x)$, $\det A_n(x) = J_{p,n}(x)$ for $n > 2$.

$$\det A_{n+1}(x) = p(x)\det A_n(x) - 2i^2 \det A_{n-1}(x) = p(x)J_{p,n}(x) + 2J_{p,n-1}(x).$$

■

Theorem 15 *Assume that* $B_n(x)$ *is a nxn tridigional matrix defined as*

$$B_n(x) = \begin{bmatrix} 2 & 2i & & & & \\ 0 & p(x)/2 & 2i & . & & \\ & i & p(x) & . & & \\ & & . & . & . & \\ & & & . & . & . \\ & & & & . & . & 2i \\ & & & & & i & p(x) \end{bmatrix}$$

where $i=\sqrt{-1}$ and $n \geq 0$. Then

$$\det B_n(x) = C_{p,n-1}(x).$$

Proof. The proof is obtaied by using the proof of Theorem 14. ■

III. MATRIX FORM OF $p(x)$–*JACOBSTHAL AND* $p(x)$–*JACOBSTHAL LUCAS* POLYNOMIAL *SEQUENCES*

We demonstrate that the matrix

$$J_p = \begin{bmatrix} p(x) & 2 \\ 1 & 0 \end{bmatrix} \tag{3.1}$$

generates $p(x)$-*Jacobsthal* polynomials and $p(x)$-*Jacobsthal Lucas* polynomials, use it to deduce some identities of these polynomials.

Theorem 16 *Let n is a positive integer. Then*

$$J_p^n = \begin{bmatrix} J_{p,n+1}(x) & 2J_{p,n}(x) \\ J_{p,n}(x) & 2J_{p,n-1}(x) \end{bmatrix} \tag{3.2}$$

Proof. We use induction method for proof. We can easily see the assertion is true for $n=1$. Assume that the statement is true for $m \leq n$. We want to show the result is also true for $n+1$.

$$J_p^{n+1} = J_p^n J_p = \begin{bmatrix} J_{p,n+1}(x) & 2J_{p,n}(x) \\ J_{p,n}(x) & 2J_{p,n-1}(x) \end{bmatrix} \begin{bmatrix} p(x) & 2 \\ 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} J_{p,n+2}(x) & 2J_{p,n+1}(x) \\ J_{p,n+1}(x) & 2J_{p,n}(x) \end{bmatrix}.$$

From this theorem we can write the following property for $p(x)$-*Jacobsthal* polynomials ■

$$\begin{bmatrix} J_{p,n+1}(x) \\ J_{p,n}(x) \end{bmatrix} = J_p^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{3.3}$$

We can write similar property for $p(x)$-*Jacobsthal Lucas* polynomials as the following

$$\begin{bmatrix} C_{p,n+1}(x) \\ C_{p,n}(x) \end{bmatrix} = J_p^n \begin{bmatrix} p(x) \\ 2 \end{bmatrix} \tag{3.4}$$

Corollary 17 *Let $m, n \geq 0$, then*

$$J_{p,m+n+1}(x) = J_{p,n+1}(x)J_{p,m+1}(x) + 2J_{p,n}(x)J_{p,m}(x) \tag{3.5}$$

Proof. The proof is made by using the property of $J_p^{m+n} = J_p^m . J_p^n$ and equality of matrix. ■

Corollary 18

$$J_{p,n+m}(x) = J_{p,n+1}(x)J_{p,m}(x) + 2J_{p,n}(x)J_{p,m-1}(x)$$
$$C_{p,m+n}(x) = C_{p,n}(x)J_{p,m-1}(x) + 2J_{p,m}(x)C_{p,n+1}(x)$$

Proof. We also can see the truth of the relation by using the Corollory 17. We want to use another method for the proof by using (3.3)

$$\begin{aligned}
&J_{p,n+1}(x)J_{p,m}(x) + 2J_{p,n}(x)J_{p,m-1}(x) \\
&= \begin{bmatrix} J_{p,m}(x) & 2J_{p,m-1}(x) \end{bmatrix} \begin{bmatrix} J_{p,n+1}(x) \\ J_{p,n}(x) \end{bmatrix} \\
&= \begin{bmatrix} J_{p,m}(x) & 2J_{p,m-1}(x) \end{bmatrix} J_p^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} J_p^{m+n-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= J_{p,m+n}(x)
\end{aligned}$$

and

$$\begin{aligned}
&J_{p,m}(x)C_{p,n+1}(x) + 2C_{p,n}(x)J_{p,m-1}(x) \\
&= \begin{bmatrix} J_{p,m}(x) & 2J_{p,m-1}(x \end{bmatrix} \begin{bmatrix} C_{p,n+1}(x) \\ C_{p,n}(x) \end{bmatrix} \\
&= \begin{bmatrix} J_{p,m}(x) & 2J_{p,m-1}(x \end{bmatrix} J_p^n \begin{bmatrix} p(x) \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} J_p^{m+n-1} \begin{bmatrix} p(x) \\ 2 \end{bmatrix} \\
&= C_{p,m+n}(x)
\end{aligned}$$

■

Corollary 19 *Let $n \geq 0$, then*

$$J_{p,n+1}(x)J_{p,n-1}(x) - J_{p,n}^2(x) = -(-2)^{n-1}$$

Proof. $\det(J_p) = -2$. Then $\det(J_p^n) = (-2)^n = 2(J_{p,n+1}(x)J_{p,n-1}(x) - J_{p,n}^2(x))$. ■

Theorem 20 *The inverse of* J_p^n

$$J_p^{-n}=\begin{bmatrix} J_{p,n+1}(x) & 2J_{p,n}(x)\\ J_{p,n}(x) & 2J_{p,n-1}(x)\end{bmatrix} \tag{3.6}$$

Theorem 21 *The eigenvalues of* J_p^n *are* $\alpha^n(x)$ *and* $B^n(x)$.

Proof.

$$\begin{aligned}
&\det(J_p^n-\lambda I)\\
&=\det\begin{bmatrix} J_{p,n+1}(x)-\lambda & 2J_{p,n}(x)\\ J_{p,n}(x) & 2J_{p,n-1}(x)-\lambda\end{bmatrix}=0\\
&=\lambda^2-\lambda(J_{p,n+1}(x)+2J_{p,n-1}(x))+2J_{p,n+1}(x)J_{p,n-1}(x)-2J_{p,n}^2(x)\\
&=\lambda^2-\lambda C_{p,n}(x)+(-2)^n
\end{aligned}$$

The roots are found from the important relationships in f) and g)

$$\alpha^n(x)=\frac{C_{p,n}(x)+\sqrt{p^2(x)+8}J_{p,n}(x)}{2}$$
$$\beta^n(x)=\frac{C_{p,n}(x)-\sqrt{p^2(x)+8}J_{p,n}(x)}{2}$$

■

Corollary 22 *We offer two relationships that can be described as being of the* Moivre type *by using the important relationships f) , g)*

$$\begin{aligned}
\left[\sqrt{p^2(x)+8}J_{p,n}(x)+C_{p,n}(x)\right]^r&=2^r\alpha^{nr}\\
&=2^{r-1}\left[\sqrt{p^2(x)+8}J_{p,nr}(x)+C_{p,nr}(x)\right]\\
\left[\sqrt{p^2(x)+8}J_{p,n}(x)-C_{p,n}(x)\right]^r&=(-2)^r\beta^{nr}\\
&=(-2)^{r-1}\left[\sqrt{p^2(x)+8}J_{p,nr}(x)-C_{p,nr}(x)\right].
\end{aligned}$$

When $p(x)=1$ this property reduces to the following form

$$\left[\frac{3J_n+C_n}{2}\right]^r=\frac{3J_{nr}+C_{nr}}{2},$$
$$\left[\frac{3J_n-C_n}{2}\right]^r=\frac{C_{nr}-3J_{nr}}{2}.$$

Acknowledgement I'm grateful to the anonymous referees for their valuable comments and suggestion for construction of the manuscript.

---

[1] T. Koshy, Fibonacci and Lucas Numbers with Applications, Toronto, New York, NY, USA, 2001.

[2] A.F. Horadam and Bro. J.M. Mahon, Pell and Pell–Lucas Polynomials, Fibonacci Quarterly 23(1), (1985), 7–20.

[3] A . F. Horadam, Jacobsthal Representation Numbers, The Fibonacci Quarterly, 34(1), 1996, 40-54.

[4] A. F. Horadam, Jacobsthal Representation Polynomials, The Fibonacci Quarterly, 35(2), (1997), 137-148.

[5] A. F. Horadam, P. Flipponi, Derivative Sequences of Jacobsthal and Jacobsthal LucasPolynomials, The Fibonacci Quarterly, 35(1), (1997), 352-357.

[6] S. Falcon, A. Plaza, On the k-Fibonacci numbers, Chaos Solitons and Fractals, 32(5), (2007), 1615-1624.

[7] S. Uygun, The (s,t)-Jacobsthal and (s,t)–Jacobsthal Lucas sequences, Applied Mathematical Sciences, 70(9), (2015), 3467-3476.

[8] B. Singh, P. Bhadouria, O. Sikhwal, Generalized Identities Involving Common Factors Fibonacci and Lucas numbers, International Journal of Algebra, Vol. 15, No. 13, (2011), 637-645.

[9] A. Nalli, P. Haukkanen, On Generalized Fibonacci and Lucas polynomials, Chaos, Solitons and Fractals, 42, (2009), 3179-3186.

[10] M.N. S. Swammy, Generalized Fibonacci and Lucas polynomials and their associated diagonal polynomials, The Fibonacci Quarterly, 37(3), (1999), 213-222.

[11] M. Catalini, Some Formulae for Bivariate Fibonacci and Lucas polynomials, arXiv:maath.CO/0407105v1,7 Jul 2004.

[12] G. B.Djordjevic, Generating functions of the incomplete

generalized Fibonacci and generalized Lucas numbers, Fibonacci Quarterly. 42(2), (2004), 106–113.