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Some Generalized Separation Axioms of Double Topological Spaces

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In this paper, we introduce some separation axioms on double topological spaces and some relation between them.

Keywords: Double T_i^*-spaces ($T_i^{**}-$spaces), $(i = 0,1,2,3)$.

I. INTRODUCTION

Atanassov [1–4] introduced the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets. Çoker [6] generalized topological structures in intuitionistic fuzzy case. The concept of intuitionistic sets and the topology on intuitionistic sets was first given by Çoker [5, 7].

Flou set stems from linguistic considerations of Yves Gentilhomme [9] about the vocabulary of a natural language. The mathematical definition of flou sets and binary operations on it are introduced by E. E. Kerre [12].

In 2005, the suggestion of J. G. Garcia et al. [8] that double set (D-set, for short) is a more appropriate name than flou (intuitionistic) set, and double topology (DT, for short) for the flou (intuitionistic) topology. Kandil et al. [10, 11] introduced the concept of D-sets, double point (D-point, for short), double topological spaces (DTS, for short) and continuous functions between these spaces.

In this paper, we generalized some of separation axioms on DTS. Moreover, we give the relationship between them.

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II. PRELIMINARIES

In this section, we collect some definitions and theorems which will be needed in the sequel. For more details see [10, 11].

Definition II.1 *[11] Let X be a nonempty set.*

1. *A D-set $\underline{A}$ is an ordered pair $(A_1,A_2) \in P(X)\hat{\times}P(X)$ such that $A_1 \subseteq A_2$.*
2. *$D(X) = \{(A_1,A_2) \in P(X)\hat{\times}P(X), A_1 \subseteq A_2\}$ is the family of all D-sets on X.*
3. *Let $\eta_1,\eta_2 \subseteq P(X)$. The product of η_1 and η_2, denoted by $\eta_1\hat{\times}\eta_2$, defined by: $\eta_1\hat{\times}\eta_2 = \{(A_1,A_2) : A_1 \in \eta_1, A_1 \in \eta_2, A_1 \subseteq A_2\}$.*
4. *The D-set $\underline{X} = (X,X)$ is called the universal D-set.*
5. *The D-set $\underline{\emptyset} = (\emptyset,\emptyset)$ is called the empty D-set.*

Definition II.2 *[11] Let $\underline{A} = (A_1,A_2), \underline{B} = (B_1,B_2)$ and $\underline{C} = (C_1,C_2) \in D(X)$.*

1. *$\underline{A} = \underline{B} \Leftrightarrow A_1 = B_1, A_2 = B_2$.*
2. *$\underline{A} \subseteq \underline{B} \Leftrightarrow A_1 \subseteq B_1, A_2 \subseteq B_2$.*
3. *$\underline{A} \cup \underline{B} = (A_1 \cup B_1, A_2 \cup B_2)$.*

4. $\underline{A} \sqcap \underline{B} = (A_1 \cap B_1, A_2 \cap B_2)$.

5. $\underline{A}^c = (A_2^c, A_1^c)$, *where* $\underline{A}^c$ *is the complement of* $\underline{A}$.

6. $\underline{A} \setminus \underline{B} = (A_1 \setminus B_2, A_2 \setminus B_1)$.

7. $\underline{A} \setminus (\underline{B} \sqcap \underline{C}) = (\underline{A} \setminus \underline{B}) \sqcup (\underline{A} \setminus \underline{C})$.

8. *Let* $x \in X$. *The D-sets* $\underline{x}_1 = (\{x\}, \{x\})$ *and* $\underline{x}_{\frac{1}{2}} = (\emptyset, \{x\})$ *are said to be D-point in* X. *The family of all D-points, denoted by* $DP(X)$ *i.e.,* $DP(X) = \{\underline{x}_t : x \in X, t \in \{1, \frac{1}{2}\}\}$.

9. $\underline{x}_1 \underline{\in} \underline{A} \Leftrightarrow x \in A_1$ *and* $\underline{x}_{\frac{1}{2}} \underline{\in} \underline{A} \Leftrightarrow x \in A_2$.

Definition II.3 *[10] Two D-sets* $\underline{A}$ *and* $\underline{B}$ *are said to be a quasi-coincident, denoted by* $\underline{A}q\underline{B}$, *if* $A_1 \cap B_2 \neq \emptyset$ *or* $A_2 \cap B_1 \neq \emptyset$. $\underline{A}$ *is called a not quasi-coincident with* $\underline{B}$, *denoted by* $\underline{A} \not{q} \underline{B}$, *if* $A_1 \cap B_2 = \emptyset$ *and* $A_2 \cap B_1 = \emptyset$.

Theorem II.1 *[10] Let* $\underline{A}, \underline{B}, \underline{C} \in D(X)$ *and* $\underline{x}_t \in DP(X)$. *Then,*

1. $\underline{x}_t \not{q} (\underline{A} \sqcap \underline{B}) \Leftrightarrow \underline{x}_t \not{q} \underline{A}$ *or* $\underline{x}_t \not{q} \underline{B}$.

2. $\underline{A} \not{q} \underline{B}, \underline{C} \subseteq \underline{B} \Rightarrow \underline{A} \not{q} \underline{C}$.

Definition II.4 *[11] Consider two ordinary sets* X *and* Y. *Let* f *be a mapping from* X *into* Y. *The image of a D-set* $\underline{A}$ *in* $D(X)$ *defined by:* $f(\underline{A}) = (f(A_1), f(A_2))$. *Also the inverse image of a D-set* $\underline{B} \in D(Y)$ *defined by:* $f^{-1}(\underline{B}) = (f^{-1}(B_1), f^{-1}(B_2))$.

Proposition II.1 *[11] Let* $f : X \to Y, \underline{A} \in D(X), \underline{B} \in D(Y)$. *Then,*

1. $f^{-1}(\underline{B}^c) = (f^{-1}(\underline{B}))^c$.

2. $\underline{A} \subseteq f^{-1}(f(\underline{A}))$ *and equality holds if* f *is (one-one).*

Definition II.5 *[11] Let* X *be a non-empty set. The family* η *of D-sets in* X *is called a DT on* X *if it satisfies the following axioms:*

1. $\underline{\emptyset}, \underline{X} \in \eta$,

2. *If* $\underline{A}, \underline{B} \in \eta$, *then* $\underline{A} \sqcap \underline{B} \in \eta$,

3. *If* $\{\underline{A}_s : s \in S\} \subseteq \eta$, *then* $\sqcup_{s \in S} \underline{A}_s \in \eta$.

The pair (X, η) *is called a DTS. Each element of* η *is called an open D-set in* X. *The complement of open D-set is called closed D-set.*

If η contains $(\emptyset, X)$, then the DTS (X, η) is called stratifiable double topological space, (STDTS, for short).

Definition II.6 *[11] Let* (X, η) *be a DTS. A D-set* $\underline{A} \in \underline{X}$ *is called a double neighborhood (D−nbd, for short) of the D-point* $\underline{x}_t$ $(t \in \{1, \frac{1}{2}\})$, *if there exists* $\underline{O}_{\underline{x}_t} \in \eta$ *such that* $\underline{x}_t \underline{\in} \underline{O}_{\underline{x}_t} \subseteq \underline{A}$. *The family of all D−nbds of the D-point* $\underline{x}_t$ *will be denoted by* $\underline{N}(\underline{x}_t)$.

Definition II.7 *[11] Let* (X, η) *be a DTS and* $\underline{A} \in D(X)$. *The double closure of* $\underline{A}$, *denoted by* $cl_\eta(\underline{A})$ *or* $\overline{\underline{A}}$, *defined by:* $cl_\eta(\underline{A}) = \sqcap\{\underline{B} : \underline{B} \in \eta^c$ *and* $\underline{A} \subseteq \underline{B}\}$.

Definition II.8 *[11] Let* (X, η) *be a DTS and* $\underline{A} \in D(X)$. *The double interior of* $\underline{A}$, *denoted by* $int_\eta(\underline{A})$ *or* $\underline{A}^o$, *defined by:* $int_\eta(\underline{A}) = \sqcup\{\underline{B} : \underline{B} \in \eta$ *and* $\underline{B} \subseteq \underline{A}\}$.

Definition II.9 *[11] Let* $\underline{A} = (A_1, A_2) \in D(X)$. $\underline{A}$ *is called a finite D-set if* A_2 *is a finite set.*

Definition II.10 *[11] Let* X *be an infinite set. The family* $\eta_\infty = \{\underline{\emptyset}\} \bigsqcup \{\underline{A} \subseteq \underline{X} : \underline{A}^c$ *is finite* $\}$ *is called a co-finite DT on* X.

Definition II.11 *[11] Let* (X, η) *be a DTS and* Y *be a non-empty subset of* X. $\eta_Y = \{\underline{A} \bigsqcap \underline{Y} : \underline{A} \in \eta$ *and* $\underline{Y} = (Y, Y)\}$ *is a DT on* Y. *The DTS* (Y, η_Y) *is called a double topological subspace (DT−subspace, for short) of* (X, η).

Definition II.12 *[11] Let* (X, η) *be a DTS,* $\underline{F} \in D(X)$ *and* Y *be a non-empty subset of* X. *The D-subset over* Y, *denoted by* F^Y, *defined by:* $F^Y = \underline{F} \bigsqcap \underline{Y}$.

Definition II.13 *[11] Let* $f : X \to Y$ *be a mapping and let* (X, η) *and* (Y, η^*) *be DTS.* f *is called a D-continuous if* $f^{-1}(\underline{B}) \in \eta$, *whenever* $\underline{B} \in \eta^*$.

Theorem II.2 *[11] Let* (X, η) *and* (Y, η^*) *be two D-topological spaces and let* $f : X \to Y$ *be a mapping,* $\underline{A} \in D(X)$ *and* $\underline{B} \in D(Y)$. *Then, the following conditions are equivalent:*

1. f *is a D-continuous,*

2. $f^{-1}(\underline{B}) \in \eta^c$, $\forall \underline{B} \in \eta^{*c}$,

3. $f(cl_\eta(\underline{A})) \subseteq cl_{\eta^*}(f(\underline{A}))$, $\forall \underline{A} \in D(X)$.

Definition II.14 *[11] Let (X,η) and (Y,η^*) be two DTS and let $f: X \to Y$ be a mapping and $\underline{A} \in D(X)$.*

1. *f is called D-open if $f(\underline{A}) \in \eta^*$, $\forall \underline{A} \in \eta$.*
2. *f is called D-closed if $f(\underline{A}) \in \eta^{*c}$, $\forall \underline{A} \in \eta^c$.*

Theorem II.3 *[11] Let (X,η) and (Y,η^*) be two DTS and let $f: X \to Y$ be a mapping and $\underline{A} \in D(X)$. Then, f is D-closed iff $cl_{\eta^*}(f(\underline{A})) \subseteq f(cl_\eta(\underline{A}))$, $\forall \underline{A} \in D(X)$.*

III. D-SEPARATION AXIOMS

Definition III.1 *A DTS (X,η) is called:*

1. *DT_0-space if $\underline{x}_t \not q\, \underline{y}_r \Rightarrow cl_\eta(\underline{x}_t) \not q\, \underline{y}_r$ or $cl_\eta(\underline{y}_r) \not q\, \underline{x}_t$, $\forall \underline{x}_t, \underline{y}_r \in DP(X)$. [11]*
2. *DT_1-space if $\underline{x}_t \not q\, \underline{y}_r \Rightarrow cl_\eta(\underline{x}_t) \not q\, \underline{y}_r$ and $cl_\eta(\underline{y}_r) \not q\, \underline{x}_t$, $\forall \underline{x}_t, \underline{y}_r \in DP(X)$. [11]*
3. *DT_2-space if $\underline{x}_t \not q\, \underline{y}_r \Rightarrow \exists \underline{O}_{\underline{x}_t}, \underline{O}_{\underline{y}_r} \in \eta$ such that $\underline{O}_{\underline{x}_t} \not q\, \underline{O}_{\underline{y}_r}$, $\forall \underline{x}_t, \underline{y}_r \in DP(X)$. [11]*
4. *DR_2-space if $\underline{x}_t \not q\, \underline{F}, \underline{F} \in \eta^c \Rightarrow \exists \underline{O}_{\underline{x}_t}, \underline{O}_{\underline{F}} \in \eta$ such that $\underline{O}_{\underline{x}_t} \not q\, \underline{O}_{\underline{F}}$, $\forall \underline{x}_t \in DP(X)$. [11]*
5. *DT_3-space if it is DR_2 and DT_1-spaces. [11]*
6. *DT_0^*-space if $\forall \underline{x}_t, \underline{y}_r \in DP(X), x \neq y$ we have: $\underline{x}_t \not q\, \underline{y}_r \Rightarrow cl_\eta(\underline{x}_t) \not q\, \underline{y}_r$ or $cl_\eta(\underline{y}_r) \not q\, \underline{x}_t$.*
7. *$DT_0^{**}-$space if $\forall \underline{x}_t, \underline{y}_r \in DP(X), x = y$ we have: $\underline{x}_t \not q\, \underline{y}_r \Rightarrow cl_\eta(\underline{x}_t) \not q\, \underline{y}_r$ or $cl_\eta(\underline{y}_r) \not q\, \underline{x}_t$.*
8. *DT_1^*-space if $\forall \underline{x}_t, \underline{y}_r \in DP(X), x \neq y$ we have: $\underline{x}_t \not q\, \underline{y}_r \Rightarrow cl_\eta(\underline{x}_t) \not q\, \underline{y}_r$ and $cl_\eta(\underline{y}_r) \not q\, \underline{x}_t$.*
9. *$DT_1^{**}-$space if $\forall \underline{x}_t, \underline{y}_r \in DP(X), x = y$ we have: $\underline{x}_t \not q\, \underline{y}_r \Rightarrow cl_\eta(\underline{x}_t) \not q\, \underline{y}_r$ and $cl_\eta(\underline{y}_r) \not q\, \underline{x}_t$.*
10. *DT_2^*-space if $\forall \underline{x}_t, \underline{y}_r \in DP(X), x \neq y$ and $\underline{x}_t \not q\, \underline{y}_r$ there exist $\underline{O}_{\underline{x}_t}, \underline{O}_{\underline{y}_r} \in \eta$ such that $\underline{O}_{\underline{x}_t} \not q\, \underline{O}_{\underline{y}_r}$.*
11. *$DT_2^{**}-$space if $\forall \underline{x}_t, \underline{y}_r \in DP(X), x = y$ and $\underline{x}_t \not q\, \underline{y}_r$ there exist $\underline{O}_{\underline{x}_t}, \underline{O}_{\underline{y}_r} \in \eta$ such that $\underline{O}_{\underline{x}_t} \not q\, \underline{O}_{\underline{y}_r}$.*
12. *DT_3^*-space if it is DR_2 and DT_1^*-spaces.*
13. *$DT_3^{**}-$space if it is DR_2 and $DT_1^{**}-$spaces.*

Theorem III.1 *Let (X,η) be a DTS. Then, (X,η) is DT_1-space (DT_1^*-space) iff $\underline{x}_t \not q\, \underline{y}_r, x \neq y$, implies $\exists\, \underline{O}_{\underline{x}_t}$ such that $\underline{y}_r \not q\, \underline{O}_{\underline{x}_t}$ and $\exists\, \underline{O}_{\underline{y}_r}$ such that $\underline{x}_t \not q\, \underline{O}_{\underline{y}_r}$.*

Proof. *Suppose that (X,η) is a DT_1 (DT_1^*) and let $\underline{x}_t \not q\, \underline{y}_r$. Then, $\underline{x}_t \not q\, cl_\eta(\underline{y}_r)$. Therefore, $\underline{x}_t \underline{\in} (cl_\eta(\underline{y}_r))^c \not q\, \underline{y}_r$ [by theorem II.1]. Similarly, $\underline{y}_r \underline{\in} (cl_\eta(\underline{x}_t))^c, (cl_\eta(\underline{x}_t))^c \not q\, \underline{x}_t$. Hence, the theorem holds.*

Theorem III.2 *Let (X,η) be a DTS. Then, (X,η) is a DT_0-space $\to (X,\eta)$ is a DT_0^*.*

Proof. *It is obvious.*

The following Example shows that the converse of Theorem III.2 is not true in general.

Example III.1 *Let $X = \{a,b,c\}$ and $\eta = \{\underline{\emptyset}, \underline{X}, (\{a\},\{a\}), (\{b\},\{b\}), (\{a,b\},\{a,b\})\}$.*
Then, $(X,\eta) \in DT_0^*-$space. But it is not DT_0-space, for $(\emptyset,\{a\}) \not q (\emptyset,\{a\})$, but $cl_\eta(\emptyset,\{a\}) = (\{a,c\},\{a,c\})\ q\ (\emptyset,\{a\})$.

Theorem III.3 *Let (X,η) be a DTS. Then, (X,η) is a DT_1-space $\to (X,\eta)$ is a DT_1^*.*

Proof. *It is obvious.*

The following Example shows that the converse of Theorem III.3 is not true in general.

Example III.2 *Let $X = \{a,b\}$ and $\eta = \{\underline{\emptyset}, \underline{X}, (\{a\},\{a\}), (\{a\},X), (\emptyset,\{b\}), (\{b\},\{b\})\}$.*
Then, $(X,\eta) \in DT_1^*-$space. But it is not DT_1-space, for $(\emptyset,\{a\}) \not q (\emptyset,\{a\})$, but $cl_\eta(\emptyset,\{a\}) = (\{a\},\{a\})\ q\ (\emptyset,\{a\})$.

Theorem III.4 *Let (X,η) be a DTS. Then, (X,η) is a DT_2-space $\rightarrow (X,\eta)$ is a DT_2^*.*

Proof. *It is obvious.*

The following Example shows that the converse of Theorem III.4 is not true in general.

Example III.3 *Let $X=\{a,b\}$ and $\eta=\{\underline{\emptyset},\underline{X},(\{a\},\{a\}),(\{b\},\{b\})\}$. Then, $(X,\eta)\in DT_2^*-$space. But it is not DT_2-space, for $(\emptyset,\{a\})\not q(\emptyset,\{a\})$, but $\forall\ \underline{O}_{\underline{a}_{\frac{1}{2}}},\underline{O}_{\underline{a}_{\frac{1}{2}}}\ q$ $\underline{O}_{\underline{a}_{\frac{1}{2}}},\underline{a}_{\frac{1}{2}}=(\emptyset,\{a\})$.*

Theorem III.5 *Let (X,η) be a DTS. Then, (X,η) is a DT_3-space $\rightarrow (X,\eta)$ is a DT_3^*.*

Proof. *It is obvious.*

The following Example shows that the converse of Theorem III.5 is not true in general.

Example III.4 *From Example III.3, we have $(X,\eta)\in DT_3^*-$space. But it is not DT_3-space, for $(\emptyset,\{a\})\not q(\emptyset,\{a\})$, but $(\{a\},\{a\})=cl_\eta(\emptyset,\{a\})\ q\ (\emptyset,\{a\})$.*

Theorem III.6 *Let (X,η) be a DTS. Then, (X,η) is a DT_1^*-space $\rightarrow (X,\eta)$ is a DT_o^*.*

Proof. *It is obvious.*

Example III.5 *From Example III.1, we have $(X,\eta)\in DT_0^*-$space. But it is not DT_1^*-space, for $(\{a\},\{a\})\ /\ q(\{c\},\{c\})$, but $(\{a,c\},\{a,c\})=cl_\eta(\{a\},\{a\})\ q\ (\{c\},\{c\})$.*

Theorem III.7 *Let (X,η) be a DTS. Then, (X,η) is a DT_2^*-space $\rightarrow (X,\eta)$ is a DT_1^*.*

Proof. *It follows from Theorem III.1.*

Example III.6 *Let N be the set of all natural numbers. Then, the family $\eta_N=\{\underline{\emptyset}\}\bigcup\{\underline{A}\subseteq\underline{N}:\underline{A}^c\ is\ finite\ \}$, $(N,\eta)\in DT_1^*-$space. But it is not DT_2^*, for if there exist $\underline{n}_t\not q\ \underline{s}_r$, then all open D-sets contain $\underline{n}_t$ quasi coincident with all open D-sets contain $\underline{s}_r$.*

Theorem III.8 *Let (X,η) be a DTS. Then, (X,η) is a DT_3^*-space $\rightarrow (X,\eta)$ is a DT_2^*.*

Proof. *Suppose that (X,η) is a DT_3^*-space and let $\underline{x}_t\ /\ q\ \underline{y}_r, x\neq y$. Then, $\underline{x}_t\not q\ cl_\eta(\underline{y}_r)$. It follows that, $\exists\underline{O}_{cl_\eta(\underline{y}_r)}\in\underline{N}(cl_\eta(\underline{y}_r)),\underline{O}_{\underline{x}_t}\in\underline{N}(\underline{x}_t)$ such that $\underline{O}_{cl_\eta(\underline{y}_r)}\not q\ \underline{O}_{\underline{x}_t}$. This implies that, $\underline{O}_{\underline{y}_r}\not q\ \underline{O}_{\underline{x}_t}$ [by theorem II.1]. Hence, (X,η) is a DT_2^*.*

Remark III.1 *1. From Example III.6, (X,η) is a DT_1-space, but it is not DT_2^* and from Example III.2, (X,η) is a DT_2^*-space, but it is not DT_1.*

2. From Example III.3, (X,η) is a DT_3^*-space, but it is not DT_2.

Remark III.2 *Let (X,η) be a DTS. Then,*

1. *DT_i^* is DT_i, $(i=0,1,3)$ iff $\forall x\in X,\underline{x}_{\frac{1}{2}}\not q\ cl_\eta(\underline{x}_{\frac{1}{2}})$*
2. *DT_2^* is DT_2 iff $\forall x\in X,\ \exists\ \underline{O}_{\underline{x}_{\frac{1}{2}}}\not q\ \underline{O}_{\underline{x}_{\frac{1}{2}}}$.*

Remark III.3 *Theorems III.1, III.2, III.3, III.4, III.5, III.6, III.7, III.8 are satisfied if we replace DT_i^* by DT_i^{**}, $(i=0,1,2,3)$.*

Corollary III.1 *For a DTS (X,η) we have the following implication:*

$$
\begin{array}{ccccccc}
DT_3^* & \rightarrow & DT_2^* & \rightarrow & DT_1^* & \rightarrow & DT_0^*. \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
DT_3 & \rightarrow & DT_2 & \rightarrow & DT_1 & \rightarrow & DT_0. \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
DT_3^{**} & \rightarrow & DT_2^{**} & \rightarrow & DT_1^{**} & \rightarrow & DT_0^{**}.
\end{array}
$$

IV. D-SUBSPACES

Theorem IV.1 *Let (Y,η_Y) be a $DT-$subspace of a D-space (X,η) and let $\underline{F}\in D(Y)$. Then,*

1. *If $\underline{F}$ is an open D-set in Y and $\underline{Y}\in\eta$, then $\underline{F}\in\eta$.*
2. *$\underline{F}$ is a closed D-set in Y iff $\underline{F}=\underline{Y}\cap\underline{H}$ for some $\underline{H}\in\eta^c$.*

Proof.

1. *Let $\underline{F}\in\eta_Y$. Then, $\exists\underline{G}\in\eta$ such that $\underline{F}=\underline{Y}\cap\underline{G}$.*
   Now, if $\underline{Y}\in\eta$. Then $\underline{Y}\cap\underline{G}\in\eta$. Hence, $\underline{F}\in\eta$.

2. *Let* $\underline{F} \in \eta_Y^c$. *Then,* $\underline{F} = \underline{Y} \setminus \underline{G}, \underline{G} \in \eta_Y$ *and* $\underline{G} = \underline{Y} \sqcap \underline{H}$ *for some* $\underline{H} \in \eta$. *It follows that,* $\underline{F} = \underline{Y} \setminus (\underline{Y} \sqcap \underline{H}) = \underline{Y} \setminus \underline{H} = \underline{Y} \sqcap \underline{H}^c$, *where* $\underline{H}^c$ *is a closed D-set in* X.
   Conversely, suppose that $\underline{F} = \underline{Y} \sqcap \underline{G}$ *for some* $\underline{G} \in \eta^c$, *then*
$$\begin{aligned} \underline{F} &= \underline{Y} \sqcap \underline{G} \\ &= \underline{Y} \sqcap (\underline{X} \setminus \underline{H}), (\underline{G} = \underline{X} \setminus \underline{H}, \underline{H} \in \eta) \\ &= \underline{Y} \sqcap \underline{H}^c \\ &= \underline{Y} \setminus \underline{H} \\ &= \underline{Y} \setminus (\underline{Y} \sqcap \underline{H}), \underline{Y} \sqcap \underline{H} \in \eta_Y. \end{aligned}$$
   Therefore, $\underline{F} \in \eta_Y^c$. *Hence, the result.*

Theorem IV.2 *Let* (Y, η_Y) *be a* $DT-$*subspace of a* DTS (X, η) *and let* $\underline{N}^Y \in D(Y)$. *Then, if* $\underline{N}^Y = \underline{Y} \sqcap \underline{N}$ *for some* $\underline{N} \in \underline{N}(y_r)$, *then* $\underline{N}^Y \in \underline{N}^Y(y_r)$.

Proof.
Let $\underline{N}^Y = \underline{Y} \sqcap \underline{N}, \underline{N} \in \underline{N}(y_r)$. *Then,* $\exists\ \underline{G} \in \eta$ *such that* $y_r \underline{\in} \underline{G} \subseteq \underline{N}$, *so that* $y_r \underline{\in}\ \underline{G} \sqcap \underline{Y} \subseteq \underline{N} \sqcap \underline{Y} = \underline{N}^Y$. *Therefore,* $y_r \underline{\in}\ \underline{G}^Y \subseteq \underline{N}^Y, (\underline{G}^Y = \underline{G} \sqcap \underline{Y})$. *Hence,* $\underline{N}^Y \in \underline{N}^Y(y_r)$.

Theorem IV.3 *A* $DT-$*subspace* (Y, η_Y) *of a* DT_0^*-*space* (X, η) *is a* DT_0^*.

Proof. *Let* $\underline{x}_t, \underline{y}_r \in DP(Y), x \neq y$ *such that* $\underline{x}_t \not q\ \underline{y}_r$. *Then,* $\underline{x}_t, \underline{y}_r \in DP(X)$ *and* $\underline{x}_t \not q\ \underline{y}_r$. *This implies that,* $\underline{x}_t \not q\ cl_\eta(\underline{y}_r)$ *or* $\underline{y}_r \not q\ cl_\eta(\underline{x}_t)$. *Thus,* $(\underline{x}_t \sqcap \underline{Y}) \not q\ (cl_\eta(\underline{y}_r) \sqcap \underline{Y})$ *or* $(\underline{y}_r \sqcap \underline{Y}) \not q\ (cl_\eta(\underline{x}_t) \sqcap \underline{Y})$ *[by Theorem IV.1]. Therefore,* $\underline{x}_t \not q\ cl_{\eta_Y}(\underline{y}_r)$ *or* $\underline{y}_r \not q\ cl\eta_Y(\underline{x}_t)$. *Hence,* (Y, η_Y) *is a* DT_0^*-*space.*

Theorem IV.4 *A* $DT-$*subspace* (Y, η_Y) *of a* DT_1^*-*space* (X, η) *is a* DT_1^*.

Proof. *Let* $\underline{x}_t, \underline{y}_r \in DP(Y), x \neq y$ *such that* $\underline{x}_t \not q\ \underline{y}_r$. *Then,* $\underline{x}_t, \underline{y}_r \in DP(X)$ *and* $\underline{x}_t \not q\ \underline{y}_r$ *implies* $\underline{x}_t \not q\ cl_\eta(\underline{y}_r)$ *and* $\underline{y}_r \not q\ cl_\eta(\underline{x}_t)$. *Thus* $(\underline{x}_t \sqcap \underline{Y}) \not q\ (cl_\eta(\underline{y}_r) \sqcap \underline{Y})$ *and* $(\underline{y}_r \sqcap \underline{Y}) \not q\ (cl_\eta(\underline{x}_t) \sqcap \underline{Y})$ *[by Theorem IV.1]. Therefore,* $\underline{x}_t \not q\ cl_{\eta_Y}(\underline{y}_r)$ *and* $\underline{y}_r \not q\ cl_{\eta_Y}(\underline{x}_t)$. *Hence,* (Y, η_Y) *is a* DT_1^*-*space.*

Theorem IV.5 *A* $DT-$*subspace* (Y, η_Y) *of a* DT_2^*-*space* (X, η) *is a* DT_2^*.

Proof. *Let* $\underline{x}_t, \underline{y}_r \in DP(Y), x \neq y$ *such that* $\underline{x}_t \not q\ \underline{y}_r$. *Then,* $\exists \underline{O}_{\underline{x}_t}, \underline{O}_{\underline{y}_r} \in \eta$ *such that* $\underline{O}_{\underline{x}_t} \not q\ \underline{O}_{\underline{y}_r}$, *implies* $\underline{O}_{\underline{x}_t} \sqcap \underline{Y}, \underline{O}_{\underline{y}_r} \sqcap \underline{Y} \in \eta_Y$ *[by Theorem IV.2] such that* $\underline{O}_{\underline{x}_t} \sqcap \underline{Y} \not q\ \underline{O}_{\underline{y}_r} \sqcap \underline{Y}$. *Hence,* (Y, η_Y) *is a* DT_2^*-*space.*

Theorem IV.6 *A* $DT-$*subspace* (Y, η_Y) *of a* DT_3^*-*space* (X, η) *is a* DT_3^*.

Proof. *Since* (X, η) *is a* DT_3^*-*space, then it is* DR_2 *and* DT_1^*-*spaces.*
Let $\underline{y}_r \in DP(Y)$ *and* $\underline{y}_r \not q\ \underline{F} \sqcap \underline{Y}, \underline{F} \in \eta^c$. *Then,* $\underline{y}_r \not q\ \underline{F}$ *[by theorem II.1] implies* $\exists \underline{O}_{\underline{y}_r}, \underline{O}_{\underline{F}} \in \eta$ *such that* $\underline{O}_{\underline{y}_r} \not q\ \underline{O}_{\underline{F}}$. *It follows that* $\underline{O}^Y_{\underline{y}_r} = \underline{O}_{\underline{y}_r} \sqcap \underline{Y} \not q\ \underline{O}_{\underline{F}} \sqcap \underline{Y} = \underline{O}^Y_{\underline{F}}, (\underline{O}^Y_{\underline{y}_r}, \underline{O}^Y_{\underline{F}} \in \eta_Y)$ *[by Theorem IV.2]. Therefore,* (Y, η_Y) *is a* DR_2. *But* (Y, η_Y) *is a* DT_1^*-*space [by theorem IV.4]. Hence,* (Y, η_Y) *is a* DT_3^*-*space.*

Theorem IV.7 *A D-subspace* (Y, η_Y) *of a* $DT_i^{**}-$*space* (X, η) *is a* $DT_i^{**}-$*space, i=(0, 1, 2, 3).*

Proof. *It is obvious.*

V. SOME PROPERTIES OF D-CONTINUOUS FUNCTION

Definition V.1 *Let* (X, η) *and* (Y, η^*) *be two* DTS *and let* $f : X \to Y$ *be a mapping. Then,* f *is called a D-homeomorphism if it is a one-one, D-continuous and D-closed of* X *onto* Y.

Lemma V.1 *Let* (X, η) *and* (Y, η^*) *be two* DTS *and let* $f : X \to Y$ *be a (one-one) and onto mapping. Then,*

1. *If* $\underline{y}_r \in DP(Y)$, *then,* $\exists\ x \in X$ *such that* $\underline{x}_t \in DP(X)$ *and* $f(\underline{x}_t) = \underline{y}_r$.
2. *If* $\underline{y}_r \in DP(Y)$, *then* $f^{-1}(\underline{y}_r) \in DP(X)$.
3. *If* $\underline{y}_{1_t}, \underline{y}_{2_r} \in DP(Y)$, $\underline{y}_{1_t} \not q\ \underline{y}_{2_r}, y_1 \neq y_2$ *then* $\exists\ x_1, x_2 \in X, x_1 \neq x_2$ *such that* $f(x_i) = y_i, (i = 1, 2)$ *and* $f(\underline{x}_{1_t}) = \underline{y}_{1_t}, f(\underline{x}_{2_r}) = \underline{y}_{2_r}$. *Also,* $\underline{x}_{1_t} \not q\ \underline{x}_{2_r}$.

Proof.
(1) *and* (2) *are obvious.*
(3) *It is clear from* (2) *that* $f(\underline{x}_{1_t}) = \underline{y}_{1_t}, f(\underline{x}_{2_r}) = \underline{y}_{2_r}$. *Now, if* $y_1 \neq y_2$, *then* $f(x_1) \neq f(x_2)$. *Implies that,* $x_1 \neq x_2$. *Since,* $\underline{y}_{1_t} \not q\ \underline{y}_{2_r}$, *then* $\underline{y}_{1_t} \subseteq (\underline{y}_{2_r})^c$, *so* $f^{-1}(\underline{y}_{1_t}) \subseteq f^{-1}(\underline{y}_{2_r})^c = (f^{-1}(\underline{y}_{2_r}))^c$ *[by Proposition II.1]. Thus,* $\underline{x}_{1_t} \subseteq (\underline{x}_{2_r})^c$. *Therefore,* $\underline{x}_{1_t} \not q\ \underline{x}_{2_r}, x_1 \neq x_2$.

Proposition V.1 *Let (X,η) and (Y,η^*) be two DTS and let $f : X \to Y$ be a (one-one) and onto mapping, $\underline{A} \in D(X)$. Then, $(f(\underline{A}))^c = f((\underline{A})^c)$.*

Proof. *Suppose that f is (one-one) and onto mapping, then $\underline{A} = f^{-1}(f(\underline{A}))$ [by Proposition II.1], implies that $(\underline{A})^c = (f^{-1}(f(\underline{A})))^c = f^{-1}(f(\underline{A}))^c)$ [by Proposition II.1], so that $f((\underline{A})^c) = f(f^{-1}(f(\underline{A}))^c) = (f(\underline{A}))^c$. Hence, $(f(\underline{A}))^c = f((\underline{A})^c)$.*

Theorem V.1 *The property of being a DT_0^*-space is a topological property.*

Proof. *Suppose $f : (X,\eta) \to (Y,\eta^*)$ is a D-homeomorphism. Let $\underline{y}_{1_t}, \underline{y}_{2_r} \in DP(Y), \underline{y}_{1_t} \not q\ \underline{y}_{2_r}, y_1 \neq y_2$. Then, by lemma V.1 $\exists\ x_1, x_2 \in X, x_1 \neq x_2$ such that $f(x_i) = y_i, (i = 1,2)$ and $f(\underline{x}_{1_t}) = \underline{y}_{1_t}, f(\underline{x}_{2_r}) = \underline{y}_{2_r}$. Also, $\underline{x}_{1_t} \not q\ \underline{x}_{2_r}$, and (X,η) is a DT_0^*-space, then $cl_\eta(\underline{x}_{1_t}) \not q\ \underline{x}_{2_r}$ or $\underline{x}_{1_t} \not q\ cl_\eta(\underline{x}_{2_r})$. Implies that, $\underline{x}_{1_t} \subseteq (cl_\eta(\underline{x}_{2_r}))^c$. So that $f(\underline{x}_{1_t}) \subseteq f((cl_\eta(\underline{x}_{2_r}))^c) = (f((cl_\eta(\underline{x}_{2_r})))^c$ [by Proposition V.1]. Thus $\underline{y}_{1_t} \underline{\in} (cl_{\eta^*}(f(\underline{x}_{2_r})))^c, f$ is D-homeomorphism. It follows that, $\underline{y}_{1_t} \not\underline{\in}\ cl_{\eta^*}(\underline{y}_{2_r})$. Similarly, we also have $\underline{y}_{2_r} \not\underline{\in}\ cl_{\eta^*}(\underline{y}_{1_t})$. Hence, (Y,η^*) is a DT_0^*.*

Theorem V.2 *The property of being a DT_1^*-space is a topological property.*

Proof. *Suppose $f : (X,\eta) \to (Y,\eta^*)$ is a D-homeomorphism. Let $\underline{y}_{1_t}, \underline{y}_{2_r} \in DP(Y), \underline{y}_{1_t} \not q\ \underline{y}_{2_r}, y_1 \neq y_2$. Then, by lemma V.1 $\exists\ x_1, x_2 \in X, x_1 \neq x_2$ such that $f(x_i) = y_i, (i = 1,2)$ and $f(\underline{x}_{1_t}) = \underline{y}_{1_t}, f(\underline{x}_{2_r}) = \underline{y}_{2_r}$. Also, $\underline{x}_{1_t} \not q\ \underline{x}_{2_r}$, (X,η) is a DT_1^*-space, then $cl_\eta(\underline{x}_{1_t}) \not q\ \underline{x}_{2_r}$ and $\underline{x}_{1_t} \not q\ cl_\eta(\underline{x}_{2_r})$, implies that $\underline{x}_{1_t} \subseteq (cl_\eta(\underline{x}_{2_r}))^c$, so that $f(\underline{x}_{1_t}) \subseteq f((cl_\eta(\underline{x}_{2_r}))^c) = (f((cl_\eta(\underline{x}_{2_r})))^c$ [by Proposition V.1], thus $\underline{y}_{1_t} \underline{\in} (cl_{\eta^*}(f(\underline{x}_{2_r})))^c, f$ is D-homeomorphism. It follows that, $\underline{y}_{1_t} \not\underline{\in}\ cl_{\eta^*}(\underline{y}_{2_r})$. Similarly, we also have $\underline{y}_{2_r} \not\underline{\in}\ cl_{\eta^*}(\underline{y}_{1_t})$. Hence, (Y,η^*) is a DT_1^*.*

Theorem V.3 *The property of being a DT_2^*-space is a topological property.*

Proof. *Suppose $f : (X,\eta) \to (Y,\eta^*)$ is a D-homeomorphism. Let $\underline{y}_{1_t}, \underline{y}_{2_r} \in DP(Y), \underline{y}_{1_t} \not q\ \underline{y}_{2_r}, y_1 \neq y_2$. Then, by lemma V.1 $\exists\ x_1, x_2 \in X, x_1 \neq x_2$ such that $f(x_i) = y_i, (i = 1,2)$ and $f(\underline{x}_{1_t}) = \underline{y}_{1_t}, f(\underline{x}_{2_r}) = \underline{y}_{2_r}$. As, $\underline{x}_{1_t} \not q\ \underline{x}_{2_r}$ and (X,η) is a DT_2^*-space, $\exists\ \underline{F}, \underline{G} \in \eta$ such that $\underline{x}_{1_t} \underline{\in}\ \underline{F}, \underline{x}_{2_r} \underline{\in}\ \underline{G}$ and $\underline{F} \not q\ \underline{G}$, implies that $f(\underline{x}_{1_t}) \underline{\in} f(\underline{F}), f(\underline{x}_{2_r}) \underline{\in} f(\underline{G})$ and $f(\underline{F}) \not q\ f(\underline{G})$ [by Proposition V.1] so that $\underline{y}_{1_t} \underline{\in} f(\underline{F}), \underline{y}_{2_r} \underline{\in} f(\underline{G})$ and $f(\underline{F}) \not q\ f(\underline{G}), [f(\underline{F}), f(\underline{G}) \in \eta^*]$. Hence, (Y,η^*) is a DT_2^*.*

Theorem V.4 *The property of being a DT_3^*-space is a topological property.*

Proof. *Suppose $f : (X,\eta) \to (Y,\eta^*)$ is a D-homeomorphism and (X,η) is a DT_3^*-space, then (X,η) is DT_1^* and DR_2-spaces, implies (Y,η^*) is DT_1^* and DR_2-spaces [by theorem V.2], [11]. Hence, (Y,η^*) is a DT_3^*.*

Theorem V.5 *The property of being a $DT_i^{**}-$space, (i=0, 1, 2, 3) is a topological property.*

Proof. *Straightforward.*

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