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Generalized k - Jacobsthal Sequence

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The study is about a new generalization of the second order number sequences called generalized $k-$ Jacobsthal sequences. We investigate some properties of this sequence and in the sequel of this paper we prove some of the properties by using determinant.

Keywords: Jacobsthal numbers, Binet′s formula, Generating functions.

I. INTRODUCTION

Special integer sequences such as Fibonacci, Lucas, Jacobsthal, Pell, Horadam are important for various reasons since we can see abundant applications in Physics, Engineering, Architecture, Nature and Art. So the researchers have studied abuut them for a long time. For instance, the ratio of two consecutive elements of Fibonacci sequence is called golden ratio, You can encounter it almost every area of science and art. And specially computers use conditional directives to change the flow of execution of a program. In addition to branch instructions, some microcontrollers use skip instructions which conditionally bypass the next instruction.This brings out being useful for one case out of the four possibilities on 2 bits, 3 cases on 3 bits, 5 cases on 4 bits, 11 cases on 5 bits, 21 cases on 6 bits,..., which are exactly the Jacobsthal numbers. Jacobsthal sequence is defined by the recurrence relations $j_n = j_{n-1} + 2j_{n-2}$, $j_0 = 0$, $j_1 = 1$ for $n \geq 2$, respectively. Because of the importance of special integer sequences, the scientists generalize them by the different methods. We can see any properties of these numbers in all references of us.

In this paper, a new generalization of the Jacobsthal sequence is introduced. It should be noted that the recurrence formulas of these numbers depend on two real parameters $f(k)$ and $g(k)$. The main purpose of this paper to establish some properties of this generalized k-Jacobsthal sequence such as Binet formula, generating function, Catalan, D'ocagne...Moreever new interesting properties are revealed by using determinant of matrix whose entries are generalized k-Jacobsthal numbers. Some similar results for the generalized k-Horadam sequence are obtained by Yazlik and Taskara in [8].

II. MAIN RESULTS

In this part we define e new generalization of Jacobsthal sequences called generalized k-Jacobsthal sequences. We establish Binet formula and generating function and other different properties.

Definition 1 *For $f^2(k) + 8g(k) > 0$, let k any positive real number, $f(k)$ and $g(k)$ scalar valued polynomials, then the generalized k-Jacobsthal sequence $\{J_{k,n}\}_{n\in\mathbb{N}}$ is defined recurrently by*

$$J_{k,n} = f(k)J_{k,n-1} + 2g(k)J_{k,n-2},\ \ J_{k,0} = a,\ J_{k,1} = b,\ \ n \geq 2, \tag{1}$$

- If $f(k)=g(k)=1$ and $a=0$; $b=1$, we get the classic Jacobsthal sequence is obtained.

The first few of the elements of the generalized k-Jacobsthal sequence are $J_{k,0}=a,\ J_{k,1}=b,\ J_{k,2}=bf(k)+2ag(k),\ \ J_{k,3}=bf^2(k)+2af(k)g(k)+2bg(k)...$

The characteristic equation of (1) is given with the following formula

$$x^2-f(k)x-2g(k)=0$$

with roots

$$\alpha=\frac{f(k)+\sqrt{f^2(k)+8g(k)}}{2},\quad \beta=\frac{f(k)-\sqrt{f^2(k)+8g(k)}}{2}, \tag{2}$$

so that

$$\alpha+\beta=f(k),\quad \alpha\beta=-2g(k),\ \ \alpha-\beta=\sqrt{f^2(k)+4g(k)}. \tag{3}$$

Lemma 2 *Let α ,β be roots of the characteristic equation of (1) as defined in (2), then the following equality is satisfied:*

$$J_{k,n}=\alpha J_{k.n-1}+(J_{k,1}-\alpha J_{k,0})\beta^{n-1}. \tag{4}$$

Proof. *By usig the properties of α ,β we have*

$$J_{k,n}=f(k)J_{k,n-1}+2g(k)J_{n-2}$$
$$J_{k,n}=(\alpha+\beta)J_{k,n-1}-(\alpha\beta)J_{k,n-2}$$

$$J_{k,n}-\alpha J_{k,n-1}=\beta(J_{k,n-1}-\alpha J_{k,n-2}) \tag{5}$$

Similarly for $n-1\rightarrow n$,

$$J_{k,n-1}=f(k)J_{k,n-2}+2g(k)J_{k,n-3}$$

$$J_{k,n-1}=\alpha J_{k,n-2}+\beta J_{k,n-2}-(\alpha\beta)J_{k,n-3} \tag{6}$$

If we substitıte Eq. (6) into (5), we obtain

$$\begin{aligned}J_{k,n}-\alpha J_{k,n-1}&=\beta(\alpha J_{k,n-2}+\beta J_{k,n-2}-(\alpha\beta)J_{k,n-3}-\alpha J_{k,n-2})\\&=\beta^2(J_{k,n-2}-\alpha J_{k,n-3})\\J_{k,n}&=\alpha J_{k,n-1}+\beta^2(J_{k,n-2}-\alpha J_{k,n-3})\end{aligned}$$

By making the same reduction precedure n times at last we have

$$J_{k,n}=\alpha J_{k.n-1}+(J_{k,1}-\alpha J_{k,0})\beta^{n-1}.$$

■

Theorem 3 ***(Binet Forms):*** *Binet's formulas allow us to express the generalized k-Jacobsthal numbers in function of the roots α,β are defined by*

$$J_{k,n}=\frac{X\alpha^n-Y\beta^n}{\alpha-\beta} \tag{7}$$

where $X=b-a\beta,\quad Y=b-a\alpha.$

Proof. 1) Let divide both sides of (4) by β^n

$$J_{k,n}=\alpha J_{k.n-1}+(J_{k,1}-\alpha J_{k,0})\beta^{n-1}$$
$$\frac{J_{k,n}}{\beta^n}=\frac{\alpha}{\beta}\frac{J_{k.n-1}}{\beta^{n-1}}+\frac{\beta^{n-1}}{\beta^n}(J_{k,1}-\alpha J_{k,0})$$

If we write the first order linear difference equation by the above equality, we have

$$V_n=\frac{\alpha}{\beta}V_{n-1}+\frac{J_{k,1}-\alpha J_{k,0}}{\beta}$$

The solution of this equation is get by

$$V_n=(\frac{\alpha}{\beta})^nJ_{k,0}+\frac{J_{k,1}-\alpha J_{k,0}}{\beta}\frac{(\frac{\alpha}{\beta})^n-1}{\frac{\alpha}{\beta}-1}.$$

By the algebraic operations, we have

$$\begin{aligned}\frac{J_{k,n}}{\beta^n}&=\frac{\alpha^n}{\beta^n}J_{k,0}+\frac{J_{k,1}-\alpha J_{k,0}}{\beta}\frac{\alpha^n-\beta^n}{\beta^n(\alpha-\beta)}\beta\\J_{k,n}&=\alpha^n(J_{k,0}+\frac{J_{k,1}-\alpha J_{k,0}}{\alpha-\beta})-\beta^n(\frac{J_{k,1}-\alpha J_{k,0}}{\alpha-\beta})\\J_{k,n}&=\alpha^n(\frac{J_{k,1}-\beta J_{k,0}}{\alpha-\beta})-\beta^n(\frac{J_{k,1}-\alpha J_{k,0}}{\alpha-\beta})\end{aligned}$$

So the proof is completed. ■

Proof. 2) Let us define n. generalized k-Jacobsthal number by aid of the roots α,β as

$$J_{k,n}=c_1\alpha^n+c_2\beta^n.$$

For $n=0$, we get $J_{k,0}=c_1+c_2=a$ and for $n=1$, we get $J_{k,1}=c_1\alpha+c_2\beta=b$. From this two equalities, we have the desired result. ■

Theorem 4 ***(Generating Function)*** *The generating function of generalized k-Jacobsthal numbers is established as*

$$\sum_{i=0}^{\infty} J_{k,i}x^i = \frac{J_{k,0}+x(J_{k,1}-f(k)J_{k,0})}{1-f(k)x-2g(k)x^2}. \quad (8)$$

Proof. Let the generating function of generalized k-Jacobsthal numbers is $J(x) = J_{k,n} = J_{k,0}+xJ_{k,1}+...+x^nJ_{k,n}+...$ If we multiply $J(x)$ by $f(x)x$ and $2g(k)x^2$,then we get

$$f(x)xJ(x) = f(x)xJ_{k,n} = f(x)xJ_{k,0}+f(x)x^2J_{k,1}+...+f(x)x^{n+1}J_{k,n}+...$$

$$2g(k)x^2J(x) = g(k)x^2J_{k,0}+g(k)x^3J_{k,1}+...+g(k)x^{n+2}J_{k,n}+...$$

From the difference of three equalities, we have the desired result

$$(1-f(k)x-g(k)x^2)J_{k,n} = J_{k,0}+x(J_{k,1}-f(k)J_{k,0}).$$

■

Theorem 5

$$\sum_{k=0}^{n-1}\frac{J_k}{t^k} = \frac{1}{(\alpha-\beta)t^{n-1}}\left[\frac{X(\alpha^n-t^n)(\beta-t)-Y(\beta^n-t^n)(\alpha-t)}{t^2-2g(k)-f(k)t}\right]$$

Proof. The proof is easily seen by using the Binet formula. ■

Theorem 6 *For* $\left|\frac{\alpha}{t}\right|<1$ *,* $\left|\frac{\beta}{t}\right|<1$*, the generating function of the generalized k-Jacobsthal number with the negative power of t is computed as*

$$\sum_{i=0}^{\infty}\frac{J_i}{t^i} = \frac{t(b+at-af(k))}{t^2-f(k)t-2g(k)}.$$

Proof. We use Binet formula and the sum of geometric series for the proof:

$$\begin{aligned}
\sum_{i=0}^{\infty}\frac{J_i}{t^i} &= \lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{X\left(\frac{\alpha}{t}\right)^i-Y\left(\frac{\beta}{t}\right)^i}{\alpha-\beta}\\
&= \frac{1}{\alpha-\beta}\lim_{n\to\infty}\left[X\left(\frac{\left(\frac{\alpha}{t}\right)^n-1}{\frac{\alpha}{t}-1}\right)-Y\left(\frac{\left(\frac{\beta}{t}\right)^n-1}{\frac{\beta}{t}-1}\right)\right]\\
&= \frac{1}{\alpha-\beta}\left[\frac{Xt}{t-\alpha}-\frac{Yt}{t-\beta}\right]\\
&= \frac{t}{\alpha-\beta}\left[\frac{(b-a\beta)(t-\beta)-(b-a\alpha)(t-\alpha)}{(t-\alpha)(t-\beta)}\right]\\
&= \frac{t}{\alpha-\beta}\left[\frac{b(\alpha-\beta)+at(\alpha-\beta)-a(\alpha^2-\beta^2)}{t^2-f(k)t-2g(k)}\right]\\
&= \frac{t(b+at-af(k))}{t^2-f(k)t-2g(k)}.
\end{aligned}$$

■

Theorem 7 ***(Catalan Identity)*** *Assume that $r>0$ and integer, we compute the Catalan identity by the following formula*

$$J_{k,n+r}J_{k.n-r}-J_{k.n}^2 = -XY(-2g(k))^{n-r}\left(\frac{\alpha^r-\beta^r}{\alpha-\beta}\right)^2.$$

Proof. By using Binet form for this sequence, we have

$$\begin{aligned}
&J_{k,n+r}J_{k.n-r}-J_{k.n}^2\\
&= \frac{(X\alpha^{n-r}-Y\beta^{n-r})(X\alpha^{n+r}-Y\beta^{n+r})}{(\alpha-\beta)^2}-\frac{(X\alpha^n-Y\beta^n)^2}{(\alpha-\beta)^2}\\
&= \frac{X^2\alpha^{2n}+Y^2\beta^{2n}-XY(\alpha\beta)^n\left[\frac{\beta^r}{\alpha^r}+\frac{\alpha^r}{\beta^r}\right]}{(\alpha-\beta)^2}\\
&\quad -\frac{X^2\alpha^{2n}+Y^2\beta^{2n}-2XY(\alpha\beta)^n}{(\alpha-\beta)^2}\\
&= \frac{XY(\alpha\beta)^n\left[2-\frac{\beta^r}{\alpha^r}-\frac{\alpha^r}{\beta^r}\right]}{(\alpha-\beta)^2}\\
&= \frac{XY(\alpha\beta)^n\left[\frac{2\alpha^r\beta^r-\beta^{2r}-\alpha^{2r}}{\alpha^r\beta^r}\right]}{(\alpha-\beta)^2}\\
&= -XY(\alpha\beta)^{n-r}\frac{(\alpha^r-\beta^r)^2}{(\alpha-\beta)^2}\\
&= -XY(-2g(k))^{n-r}\left(\frac{\alpha^r-\beta^r}{\alpha-\beta}\right)^2.
\end{aligned}$$

■

Corollary 8 ***(Cassini Identity)*** *If we substitute $r=1$ at the Catalan Identity in (9), we obtain Cassini formula for generalized k-Jacobsthal sequence as the following:*

$$J_{k,n+1}J_{k.n-1}-J_{k.n}^2 = XY(-2g(k))^{n-1}b^2.$$

Theorem 9 ***(D'ocagne Property)*** *For $m,n>0$, integer, we compute the D'ocagne property for generalized k-Jacobsthal sequence by the following formula*

$$J_{k,m}J_{k,n+1}-J_{k,m+1}J_{k,n} = \frac{XYf(k)}{f^2(k)+4g(k)}\left[(-2g(k)^m(\beta^{n-m}+\alpha^{n-m})\right].$$

where $X=b-a\beta, \quad Y=b-a\alpha$.

Proof. By using Binet form for this sequence, we have

$$\begin{aligned}
&J_{k,m}J_{k,n+1}-J_{k,m+1}J_{k,n}\\
&=\frac{X\alpha^m-Y\beta^m}{\alpha-\beta}\frac{X\alpha^{n+1}-Y\beta^{n+1}}{\alpha-\beta}\\
&-\frac{X\alpha^{m+1}-Y\beta^{m+1}}{\alpha-\beta}\frac{X\alpha^n-Y\beta^n}{\alpha-\beta}\\
&=\frac{XY}{(\alpha-\beta)^2}\left[\alpha^m\beta^n(\beta+\alpha)+\beta^m\alpha^n(\alpha+\beta)\right]\\
&=\frac{XY}{(\alpha-\beta)^2}(\alpha+\beta)\left[(\alpha\beta)^m(\beta^{n-m}+\alpha^{n-m})\right]\\
&=\frac{XYf(k)}{f^2(k)+4g(k)}\left[(-2g(k))^m(\beta^{n-m}+\alpha^{n-m})\right].
\end{aligned}$$

■

Theorem 10 ***(Honsberger Property)****Assume that $m,n>0$ and integer, we establish*

$$J_{k,m+1}J_{k,n}+J_{k,m}J_{k,n-1}=\frac{1}{(\alpha-\beta)^2}\left[\begin{array}{c} X^2\alpha^{m+n+1}(\alpha^2+1)+Y^2\beta^{m+n-1}(\beta^2+1)\\ \\ +XY(2g(k)-1)(\alpha^m\beta^{n-1}+\alpha^{n-1}\beta^m)\end{array}\right].$$

Proof. The proof is similarly made by using the Binet formula. ■

Theorem 11 *Let $p>q\geq 0$, then we have the following sum property*

$$\sum_{i=0}^{n}J_{k,pi+q}=\frac{(-2g(k))^p(J_{k,pn+q}-J_{k,q-p})-J_{k,pn+p+q}+J_{k,q}}{(-2g(k))^p-\alpha^n-\beta^n+1}.$$

Proof. By using Binet formula

$$\begin{aligned}
\sum_{i=0}^{n}J_{k,pi+q}&=\sum_{i=0}^{n}\frac{X\alpha^{pi+q}-Y\beta^{pi+q}}{\alpha-\beta}\\
&=\frac{X\alpha^q}{\alpha-\beta}\sum_{i=0}^{n}\alpha^{pi}-\frac{Y\beta^q}{\alpha-\beta}\sum_{i=0}^{n}\beta^{pi}
\end{aligned}$$

By the sum of the geometric sequences

$$\sum_{i=0}^{n}J_{k,pi+q}=\frac{X\alpha^q}{\alpha-\beta}\frac{\alpha^{p(n+1)}-1}{\alpha^p-1}-\frac{Y\beta^q}{\alpha-\beta}\frac{\beta^{p(n+1)}-1}{\beta^p-1}$$

$$=\frac{1}{\alpha-\beta}\left[\begin{array}{c}\frac{X\alpha^q(\alpha^{pn}(-2g(k))^p-\alpha^{p(n+1)}-\beta^p+1)}{(-2g(k))^p-(\alpha^p+\beta^p)+1}\\ -\frac{Y\beta^q(\alpha^{pn}(-2g(k))^p-\alpha^p-\beta^{p(n+1)}+1)}{(-2g(k))^p-(\alpha^p+\beta^p)+1}\end{array}\right]$$

$$=\frac{1}{\alpha-\beta}\left[\begin{array}{c}\frac{(-2g(k))^p(X\alpha^{pn+q}-Y\beta^{pn+q})-(X\alpha^{pn+p+q}-Y\beta^{pn+p+q}))}{(-2g(k))^p-(\alpha^p+\beta^p)+1}\\ \frac{-(X\alpha^q\beta^p-Y\alpha^p\beta^q)+(X\alpha^q-Y\beta^q}{(-2g(k))^p-(\alpha^p+\beta^p)+1}\end{array}\right]$$

$$=\frac{(-2g(k))^p(J_{k,pn+q}-J_{k,q-p})-J_{k,pn+p+q}+J_{k,q}}{(-2g(k))^p-\alpha^p-\beta^p+1}.$$

■

Theorem 12 *For $n>0$, integer, the sum of the square of the elements of this sequence is given as*

$$\sum_{i=0}^{n-1}J_{k,i}^2=\frac{1}{(\alpha-\beta)^2}\left[\begin{array}{c}\frac{(\alpha\beta)^2[(X\alpha^{n-1})^2+(Y\beta^{n-1})]+(X^2+Y^2)}{4g^2(k)-f^2(k)-4g(k)+1}\\ \\ -\frac{(X^2\beta^2+Y^2\alpha^2)+(X^2\alpha^{2n}+Y^2\beta^{2n})}{4g^2(k)-f^2(k)-4g(k)+1}+\frac{2XY(-2g(k))^n-1}{2g(k)+1}\end{array}\right].$$

Proof. The proof is similarly made by using the Binet formula and sum property of geometric series. ■

Theorem 13 *For $n\geq 0$, integer, a new sum formula for generalized k-Jacobsthal sequence is given as*

$$\sum_{i=0}^{n}\binom{n}{i}f^i(k)(2g(k))^{n-i}J_i=J_{2n}.$$

Proof.

$$\sum_{i=0}^{n}\binom{n}{i}f^i(k)(2g(k))^{n-i}J_i=\sum_{i=0}^{n}\binom{n}{i}f^i(k)(2g(k))^{n-i}\frac{Xa^i-Y\beta^i}{\alpha-\beta}$$

$$=\frac{1}{\alpha-\beta}\left[\sum_{i=0}^{n}X\binom{n}{i}(f(k)\alpha)^i(2g(k))^{n-i}-\sum_{i=0}^{n}Y\binom{n}{i}(f(k)\beta)^i(2g(k))^{n-i}\right]$$

$$=\frac{1}{\alpha-\beta}\left[X(f(k)\alpha+2g(k))^n-Y(f(k)\beta+2g(k))^n\right]$$

$$=\frac{1}{\alpha-\beta}\left[X\alpha^{2n}-Y\beta^{2n}\right]=J_{2n}.$$

■

Theorem 14 *The n. element of the generalized k-Jacobsthal sequence is established by using matrix algebra as*

$$\begin{bmatrix}J_{n+1}\\ J_n\end{bmatrix}=\begin{bmatrix}f(k) & 2g(k)\\ 1 & 0\end{bmatrix}^n\begin{bmatrix}J_1\\ J_0\end{bmatrix}.$$

Proof. Let us write two equations with two unknowns X,Y

$$J_nX+J_{n-1}Y=J_{n+1}$$

$$J_{n+1}X+J_nY=J_{n+2}$$

We denote the equations matrix form as

$$A=\begin{bmatrix}J_n & J_{n-1}\\ J_{n+1} & J_n\end{bmatrix}\qquad u=\begin{bmatrix}X\\ Y\end{bmatrix}\qquad b=\begin{bmatrix}J_{n+1}\\ J_{n+2}\end{bmatrix}.$$

For the solution of it we use Cramer method, so we get the solution for Y as

$$Y=\frac{\begin{vmatrix}J_{k,n} & J_{k,n+1}\\ J_{k,n+1} & J_{k,n+2}\end{vmatrix}}{\begin{vmatrix}J_{k,n} & J_{k,n-1}\\ J_{k,n+1} & J_{k,n}\end{vmatrix}}=2g(k)$$

$$
\begin{aligned}
W_n &= J_nJ_{n+2} - J_{n+1}^2 \\
&= (J_{n+1}J_{n-1} - J_n^2)2g(k) \\
&= 2g(k)W_{n-1} \\
&= (2g(k))^2 W_{n-2} \\
&\quad . \\
&\quad . \\
&\quad . \\
&= (2g(k))^{n-1}W_1 = (2g)^{n-1}(J_2J_0 - J_1^2) \\
&= (2g(k))^{n-1}(abf(k) + 2a^2g(k) - b^2)
\end{aligned}
$$

■

Theorem 15 *Let define a 2x2 matrix* $X_n = \begin{pmatrix} J_{k,n-1} & J_{k,n} \\ J_{k,n} & J_{k,n+1} \end{pmatrix}$ *entries are generalized k-Jacobsthal numbers, then the determinant of* X_n *is* $|X_n| = (-g(k))^{n-1}(a^2g(k) + abf(k) - b^2)$.

Proof. *We use the induction method for the proof. For* $m = 1$,

$$|X_1| = \begin{vmatrix} J_{k,0} & J_{k,1} \\ J_{k,1} & J_{k,2} \end{vmatrix} = (-2g(k))^0(2a^2g(k) + abf(k) - b^2).$$

And similarly for $m = 2$

$$|X_2| = \begin{vmatrix} J_{k,1} & J_{k,2} \\ J_{k,2} & J_{k,3} \end{vmatrix} = (-2g(k))^1(2a^2g(k) + abf(k) - b^2).$$

Let the assertion is true for all $k \le m$

$$|X_m| = \begin{vmatrix} J_{k,m-1} & J_{k,m} \\ J_{k,m} & J_{k,m+1} \end{vmatrix} = (-2g(k))^{m-1}(2a^2g(k) + abf(k) - b^2).$$

Now we want to show that it is true for $k = m+1$. *If we product the first row by* $2g(k)$ *and the second row by* $f(k)$,*we have*

$$2g(k)f(k)|X_m| = \begin{vmatrix} 2g(k)J_{k,m-1} & 2g(k)J_{k,m} \\ f(k)J_{k,m} & f(k)J_{k,m+1} \end{vmatrix}.$$

If we add the second row to the first row, we have

$$
\begin{aligned}
2g(k)f(k)|X_m| &= \begin{vmatrix} J_{k,m+1} & J_{k,m+2} \\ f(k)J_{k,m} & f(k)J_{k,m+1} \end{vmatrix} = f(k)\begin{vmatrix} J_{k,m+1} & J_{k,m+2} \\ J_{k,m} & J_{k,m+1} \end{vmatrix} \\
&= -f(k)\begin{vmatrix} J_{k,m} & J_{k,m+1} \\ J_{k,m+1} & J_{k,m+2} \end{vmatrix} = -f(k)|X_{m+1}|.
\end{aligned}
$$

From this equality, we obtain

$$
\begin{aligned}
|X_{m+1}| &= -2g(k)|X_m| \\
&\quad (-2g(k))^m(2a^2g(k) + abf(k) - b^2).
\end{aligned}
$$

From this equality, we complete the proof. ■

Theorem 16 *Assume that* $Y_r = \begin{pmatrix} J_{k,n+r} & J_{k,n} \\ J_{k,n+r+1} & J_{k,n+1} \end{pmatrix}$ *be with the entries are generalized k-Jacobsthal numbers then the following properties are hold:*

- $|Y_{r+2}| = f(k)|Y_{r+1}| + 2g(k)|Y_r|$
- $|Y_r| = (-2g(k))^n(bJ_{k,r} - aJ_{k,r+1})$.

Proof.

$$
\begin{aligned}
A &= f(k)|Y_{r+1}| + 2g(k)|Y_r| \\
&= f(k)\begin{vmatrix} J_{k,n+r+1} & J_{k,n} \\ J_{k,n+r+2} & J_{k,n+1} \end{vmatrix} + 2g(k)\begin{vmatrix} J_{k,n+r} & J_{k,n} \\ J_{k,n+r+1} & J_{k,n+1} \end{vmatrix}
\end{aligned}
$$

$$
\begin{aligned}
A &= f(k)(J_{k,n+1}J_{k,n+r+1} - J_{k,n}J_{k,n+r+2}) + 2g(k)(J_{k,n+1}J_{k,n+r} - J_{k,n}J_{k,n+r+1}) \\
&= J_{k,n+1}(f(k)J_{k,n+r+1} + 2g(k)J_{k,n+r}) - J_{k,n}(f(k)J_{k,n+r+2} + 2g(k)J_{k,n+r+1}) \\
&= J_{k,n+1}J_{k,n+r+2} - J_{k,n}J_{k,n+r+3} \\
&= |Y_{r+2}|
\end{aligned}
$$

(ii) We use induction steps on r. For $r = 0$, it is easy to see that $|Y_0| = 0$. For $r = 1$,

$$
\begin{aligned}
2g(k)f(k)|X_m| &= 2g(k)f(k)\begin{vmatrix} J_{k,m-1} & J_{k,m} \\ J_{k,m} & J_{k,m+1} \end{vmatrix} = \begin{vmatrix} 2g(k)J_{k,m-1} & f(k)J_{k,m} \\ 2g(k)J_{k,m} & f(k)J_{k,m+1} \end{vmatrix} \\
&= \begin{vmatrix} f(k)J_{k,m} + 2g(k)J_{k,m-1} & f(k)J_{k,m} \\ f(k)J_{k,m+1} + 2g(k)J_{k,m} & f(k)J_{k,m+1} \end{vmatrix} \\
&= f(k)\begin{vmatrix} J_{k,m+1} & J_{k,m} \\ J_{k,m+2} & J_{k,m+1} \end{vmatrix} = f(k)|Y_1|
\end{aligned}
$$

$$
\begin{aligned}
2g(k)|X_m| &= 2g(k)(-2g(k))^{m-1}(2a^2g(k) + abf(k) - b^2) \\
|Y_1| &= (-2g(k))^m(-2a^2g(k) - abf(k) + b^2) \\
&= (-2g(k))^m(bJ_{k,1} - aJ_{k,2})
\end{aligned}
$$

Let the assertion is true for all $k \le m$, that is $|Y_r| = (-2g(k))^m(bJ_{k,r} - aJ_{k,r+1})$. Now we want to show that it is true for $k = m+1$

$$
\begin{aligned}
|Y_r| &= (-2g(k))^m(bJ_{k,r} - aJ_{k,r+1}) \\
|Y_{r+1}| &= f(k)|Y_r| - 2g(k)|Y_{r-1}| \\
&= f(k)(-2g(k))^m(bJ_{k,r} - aJ_{k,r+1}) + 2g(k)(-2g(k))^m(bJ_{k,r-1} - aJ_{k,r}) \\
&= (-2g(k))^m\{b[f(k)J_{k,r} + 2g(k)J_{k,r-1}] - a[f(k)J_{k,r+1} + 2g(k)J_{k,r}]\} \\
&= (-2g(k))^m(bJ_{k,r+1} - aJ_{k,r+2})
\end{aligned}
$$

which ends of the proof. ■

If we choose m place of $n+r$ in the theorem ,we obtain another formula for the D/ocaigne identity for the generalized k-Jacobsthal numbers.

$$\begin{vmatrix} J_{k,m} & J_{k,n} \\ J_{k,m+1} & J_{k,n+1} \end{vmatrix}$$
$$= J_{k,m}J_{k,n+1} - J_{k,m+1}J_{k,n}$$
$$= (-2g(k))^{n}(bJ_{k,m-n} - aJ_{k,m-n+1}).$$

Theorem 17 *Assume that* $Z_s = \begin{pmatrix} J_{k,n} & J_{k,n-r} \\ J_{k,n+s} & J_{k,n-r+s} \end{pmatrix}$ *be with the entries are generalized k-Jacobsthal numbers then the following properties are hold:*

$$|Z_{s+2}| = f(k)\,|Z_{s+1}| + g(k)\,|Z_s|$$

Proof.

$$f(k)\,|Z_{s+1}| + g(k)\,|Z_s| = f(k)\begin{vmatrix} J_{k,n} & J_{k,n-r} \\ J_{k,n+s+1} & J_{k,n-r+s+1} \end{vmatrix} + 2g(k)\begin{vmatrix} J_{k,n} & J_{k,n-r} \\ J_{k,n+s} & J_{k,n-r+s} \end{vmatrix}$$

$$= f(k)(J_{k,n}J_{k,n-r+s+1} - J_{k,n-r}J_{k,n+s+1}) + 2g(k)(J_{k,n}J_{k,n-r+s} - J_{k,n+s}J_{k,n-r})$$
$$= -J_{k,n-r}(f(k)J_{k,n+s+1} + 2g(k)J_{k,n+s}) + J_{k,n}(f(k)J_{k,n-r+s+1} + 2g(k)J_{k,n-r+s})$$
$$= -J_{k,n-r}J_{k,n+s+2} + J_{k,n}J_{k,n-r+s+2}$$
$$= |Z_{s+2}|$$

■

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