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Triple Pascal Sequence Spaces

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In this paper, the concept of pascal triple sequence spaces are introduced and then basic topological properties of pascal triple sequence spaces are investigated.

Keywords: Pascal; triple sequence; Pascal sequence space.

I. INTRODUCTION

The triple pascal matrix is an infinite matrix containing the binomial coefficients as its elements. There are three ways to achieve this as either an upper-triangular matrix, a lower-triangular matrix or a symmetric matrix. The 4×4 truncation of these are show below.
The triple upper triangular

$$U_4=\begin{pmatrix}1&1&1&1\\0&1&27&96\\0&0&1&500\\0&0&0&1\end{pmatrix};$$

Triple lower triangular

$$L_4=\begin{pmatrix}1&0&0&0\\1&1&0&0\\1&27&1&0\\1&96&500&1\end{pmatrix};$$

Symmetric

$$A_4=\begin{pmatrix}1&1&1&1\\1&27&500&8575\\1&96&3375&87808\\1&250&15435&592704\end{pmatrix}$$

These matrices have the pleasing relationship $A_n=L_nU_n$. It is easily seen that all three matrices have determinant 1. The elements of the symmmetric triple pascal matrix are the binomial coefficients.
(i.e) $A_{ijk}=\begin{pmatrix}r\\m\end{pmatrix}\begin{pmatrix}s\\n\end{pmatrix}\begin{pmatrix}t\\k\end{pmatrix}=\frac{r!}{m!(r-m)!}\frac{s!}{n!(n-s)!}\frac{t!}{k!(k-t)!}$,
where $r,s,t=i+j+k$ and $m=i,n=j,k=t$.
In other words

$$A_{ijk}={}_{i+j+k}C_{ijk}=\frac{(i+j+k)!}{i!j!k!}.$$

Thus the trace of A_n is given by

$$tr(A_n)=\sum_{m=0}^{r-1}\sum_{n=0}^{s-1}\sum_{k=0}^{t-1}\frac{(2m)!}{(m!)^2}\frac{(2n)!}{(n!)^2}\frac{(2k)!}{(k!)^2}$$

with the first few terms given by the sequence 1,27,729,24389, $\cdots$.
Let A_n be $n\times n\times n$ matrix whose skew diagonals are successively the rows (truncated where necessary) of pascals triangle. In general, $A_n=(a_{ijk})$, where

$$a_{ijk} = \binom{i+j+k}{i}\binom{i+j+k}{j}\binom{i+j+k}{k} \text{ for } i,j,k = 0,1,2,\cdots,n-1.$$

An possesses the factorization

$$A_n = L_n L_n^T \tag{1}$$

where L_n^T denotes the transpose of L_n. For the $[ijk]^{th}$ secton of element of this product is
=coefficient of x^{ijk} in $(1+x)^i(1+x)^j(1+x)^k$
$= a_{ijk} = \binom{i+j+k}{i}\binom{i+j+k}{j}\binom{i+j+k}{k}$.
Clearly

$$|L_n| = 1 \tag{2}$$

so that

$$|A_n| = \left|L_n L_n^T\right| = |L_n|^2 = 1$$

we observe that L_n^{-1} is simply related to L_n.
For example

$$L_4^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -27 & 1 & 0 \\ 1 & 96 & -500 & 1 \end{pmatrix};$$

and in general

$$L_n^{-1} = (-1)^{i+j-2k} I_{ijk} \tag{3}$$

In addition, 1 is an eigen value of A_n when n is odd and that if λ is an eigen value of A_n then so is λ^{-1}. These conjectures are readily verified for small values of n. In general
Let

$$P_n(\lambda) = |\lambda I_n - A_n|$$

where I_n is the $n \times n \times n$ identity matrix. Then by (1.1),(1.2) and (1.3)

$$\begin{aligned} P_n(\lambda) &= \left|\lambda\ L_n L_n^{-1} - L_n L_n^T\right| \\ &= |L_n|\left|\lambda\ L_n^{-1} - L_n^T\right| \\ &= \left|\left((-1)^{i+j-2k}\lambda\ I_{ijk} - I_{kji}\right)\right| \\ &= (-\lambda)^n\left|\left(\lambda_{kji}^{-1} I - (-1)_{ijk}^{i+j-2k} I\right)\right|. \end{aligned}$$

Multiplying odd numbered rows and columns of the matrix by -1 and transposing, we get

$$P_n(\lambda) = (-\lambda)^n\left|\left((-1)^{i+j-2k}\lambda_{ijk}^{-1} I - I_{kji}\right)\right|$$

$$P_n(\lambda) = (-\lambda)^n P_n\left(\frac{1}{\lambda}\right) \tag{4}$$

But eigen values of A_n are the roots of $P_n(\lambda) = 0$ and thus it follows from (1.4) that if λ is an eigen value of A_n then so is λ^{-1}.

A triple sequence (real or complex) can be defined as a function $X : \mathbb{N}^3 \to \mathbb{R}(\mathbb{C})$, where $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequence was introduced and investigated at the initial by *Bipan Hazarika et al. [1]*, *Sahiner et al. [12,13]*, *Esi et al. [2-9]*, *Dutta et al. [10]*,*Subramanian et al. [14-19]*, *Debnath et al. [11]*, *Velmurugan et al. [20]* and many others.

II. THE TRIPLE PASCAL MATRIX OF INVERSE AND TRIPLE PASCAL SEQUENCE SPACES

Let P denote the Pascal means defined by the Pascal matrix as is defined by

$$P = \left[P_{mnk}^{rst}\right] = \begin{cases} \binom{r}{m}\binom{s}{n}\binom{t}{k} & \text{if } 0 \le m \le r, 0 \le n \le s, 0 \le k \le t \\ 0, & \text{if } m > r, n > s, k > t;\ r,s,t,m,n,k \in \mathbb{N} \end{cases}$$

and the inverse of Pascal's matrix

$$P = \left[P^{rst}_{mnk}\right]^{-1} = \begin{cases} (-1)^{(r-m)+(s-n)+(t-k)} \begin{pmatrix} r \\ m \end{pmatrix} \begin{pmatrix} s \\ n \end{pmatrix} \begin{pmatrix} t \\ k \end{pmatrix} & \text{if } 0 \le m \le r, 0 \le n \le s, 0 \le k \le t \\ 0, if \ (m > r, n > s, k > t;\ r,s,t,m,n,k \in \mathbb{N}) \end{cases}$$

... (*)

There is some interesting properties of Pascal matrix. For example, we can form three types of matrix; symmetric, lower triangular and upper triangular; for any integer $i,\ j,k > 0$. The symmetric Pascal matrix of order $n \times n \times n$ is defined by

$$A_{ijk} = a_{ijk} = \begin{pmatrix} i+j+k \\ i \end{pmatrix} \begin{pmatrix} i+j+k \\ j \end{pmatrix} \begin{pmatrix} i+j+k \\ k \end{pmatrix} \ for\ i,j,k = 0,1,2,\cdots,n. \tag{5}$$

We can define the lower triangular Pascal matrix of order $n \times n \times n$ by

$$L_{ijk} = \left(L_{ijk}\right) = \frac{1}{(-1)^{i+j-2k} I_{ijk}}; i,j,k = 1,2,\cdots n. \tag{6}$$

and the upper triangular Pascal matrix of order $n \times n \times n$ is defined by

$$U_{ijk} = \left(U_{ijk}\right) = \frac{1}{(-1)^{k-(i+j)} I_{ijk}}; i,j,k = 1,2,\cdots n. \tag{7}$$

We know that $U_{ijk} = \left(L_{ijk}\right)^T$ for any positive integer i,j,k.

(i) Let A_{ijk} be the symmetric Pascal matrix of order $n \times n \times n$ defined by $*, L_{ijk}$ be the lower triangular Pascal matrix of order $n \times n \times n$ defined by (2.2), then $A_{ijk} = L_{ijk} U_{ijk}$ and $det\left(A_{ijk}\right) = 1$.

(ii) Let A and B be $n \times n \times n$ matrices. We say that A is similar to B if there is an invertible $n \times n \times n$ matrix P such that $P^{-1}AP = B$.

(iii) Let A_{ijk} be the symmetric Pascal matrix of order $n \times n \times n$ defined by (2.1), then A_{ijk} is similar to its inverse A^{-1}_{ijk}.

(iv) Let L_{ijk} be the lower triangular Pascal matrix of order $n \times n \times n$ defined by (2.2), then $L^{-1}_n = L^{-1}_{ijk} = (-1)^{i+j-2k} I_{ijk}$.

We wish to introduce the Pascal sequence spaces P_{Λ^3} and P_{χ^3} as the set of all sequences such that $P-$ transforms of them are in the spaces Λ^3 and χ^3, respectively, that is

$\Lambda^3_P = \eta_{mnk} = \left\{ x = (x_{mnk}) : sup_{rst} \sum_{m=0}^{r} \sum_{n=0}^{s} \sum_{k=0}^{t} \begin{pmatrix} r \\ m \end{pmatrix} \begin{pmatrix} s \\ n \end{pmatrix} \begin{pmatrix} t \\ k \end{pmatrix} |x_{mnk}|^{\frac{1}{m+n+k}} < \infty \right\}$, and $\chi^3_P = \mu_{mnk} =$
$\left\{ x = (x_{mnk}) : lim_{rst \to \infty} \sum_{m=0}^{r} \sum_{n=0}^{s} \sum_{k=0}^{t} \begin{pmatrix} r \\ m \end{pmatrix} \begin{pmatrix} s \\ n \end{pmatrix} \begin{pmatrix} t \\ k \end{pmatrix} \left((m+n+k)! \, |x_{mnk}|\right)^{\frac{1}{m+n+k}} = 0 \right\}$.

We may redefine the spaces Λ^3_P, χ^3_P as follows:$\Lambda^3_P = P^3_\Lambda, \chi^3_P = P^3_\chi$.

If λ is an normed or paranormed sequence space; then matrix domain λ_P is called an Pascal triple sequence space. We define the triple sequence $y = (y_{rst})$ which will be frequently used, as the $P-$ transform of a triple sequence $x = (x_{rst})$

ie.,

$$y_{rst} = \sum_{m=0}^{r} \sum_{n=0}^{s} \sum_{k=0}^{t} \begin{pmatrix} r \\ m \end{pmatrix} \begin{pmatrix} s \\ n \end{pmatrix} \begin{pmatrix} t \\ k \end{pmatrix} x_{mnk}, (r,s,t \in \mathbb{N}). \tag{8}$$

Pascal sequence spaces P_{Λ^3} and P_{χ^3} as the set of all sequences such that $P-$ transforms of them are in the spaces Λ^3 and χ^3, respectively, that is

It canbe shown easily that P_{χ^3} are linear and metric space by the following metric:

$d(x,y)_{P_{\chi^3}} = d(Px,Py) = sup_{mnk}\left\{((m+n+k)!\,|x_{mnk}-y_{mnk}|)^{\frac{1}{m+n+k}} : m,n,k = 1,2,3,\cdots\right\}.$

Consider a triple sequence $x = (x_{mnk})$. The $(m,n,k)^{th}$ section $x^{[m,n,k]}$ of the sequence is defined by $x^{[m,n,k]} = \sum_{i,j,q=0}^{m,n,k} x_{ijq}\Im_{ijq}$ for all $m,n,k \in \mathbb{N}$; where $\Im_{ijq}$ denotes the triple sequence whose only non zero term is a $\frac{1}{(i+j+k)!}$ in the $(i,j,k)^{th}$ place for each $i,j,k \in \mathbb{N}$.

If X is a sequence space, we give the following definitions:

(i)X' is continuous dual of X;

(ii)$X^{\alpha} = \left\{a = (a_{mnk}) : \sum_{m,n,k=1}^{\infty} |a_{mnk}x_{mnk}| < \infty,\ for\ \ each\ \ x \in X\right\};$

(iii)$X^{\beta} = \left\{a = (a_{mnk}) : \sum_{m,n,k=1}^{\infty} a_{mnk}x_{mnk}\ \ is\ \ convergent,\ \ for\ \ each\ \ x \in X\right\};$

(iv)$X^{\gamma} = \left\{a = (a_{mnk}) : sup_{m,n,k\geq 1}\left|\sum_{m,n,k=1}^{M,N,K} a_{mnk}x_{mnk}\right| < \infty,\ for\ \ each\ \ x \in X\right\};$

(v)$Let\,X\,be\,an\,FK-space \supset \phi;\ then\,X^{f} = \left\{f(\Im_{mnk}) : f \in X'\right\};$

(vi)$X^{\delta} = \left\{a = (a_{mnk}) : sup_{m,n,k} |a_{mnk}x_{mnk}|^{1/m+n+k} < \infty,\ for\ \ each\ \ x \in X\right\};$

$X^{\alpha}, X^{\beta}, X^{\gamma}$ are called $\alpha-(or\ K\ddot{o}the-Toeplitz)$dual of $X, \beta-(or\,generalized-K\ddot{o}the-Toeplitz)\,dual\,of\ \ X, \gamma-\ dual\,of\ X,\ \delta-\ dual\,of\ X\ \ respectively.X^{\alpha}$ is defined by Gupta and Kamptan [21] . It is clear that $X^{\alpha} \subset X^{\beta}$ and $X^{\alpha} \subset X^{\gamma}$, but $X^{\alpha} \subset X^{\gamma}$ does not hold.

III. MAIN RESULTS

A. Theorem

The triple sequence spaces P_{χ^3} and P_{Λ^3} are linearly isomorphic spaces Λ^3 and χ^3 respectively i.e., $P_{\Lambda^3} \cong \Lambda^3$ and $P_{\chi^3} \cong \chi^3$.

Proof: To prove the fact $P_{\chi^3} \cong \chi^3$, we should show the existence of a linear bijection between the spaces P_{χ^3} and χ^3. Consider the transformation T defined with the notation (2.4), from P_{χ^3} to χ^3. The linearity to T is clear. Further, it is trivial that $mu = 0$ whenever $T\mu = 0$ and hence T is injective.

Let $\gamma \in \chi^3$. We consider the triple sequence $\mu = (\mu_{mnk})$ as follows:

$\mu_{mnk} =$

$\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}(-1)^{(r-m)+(s-n)+(t-k)}\begin{pmatrix} r \\ m \end{pmatrix}\begin{pmatrix} s \\ n \end{pmatrix}\begin{pmatrix} t \\ k \end{pmatrix}((m+n+k)!\,|y_{mnk}|)^{\frac{1}{m+n+k}}.$

Then

$lim_{r,s,t\to\infty}(Px)_{rst} = lim_{r,s,t\to\infty}\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}\begin{pmatrix} r \\ m \end{pmatrix}\begin{pmatrix} s \\ n \end{pmatrix}\begin{pmatrix} t \\ k \end{pmatrix}\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}$

$(-1)^{(r-m)+(s-n)+(t-k)}\begin{pmatrix} r \\ m \end{pmatrix}\begin{pmatrix} s \\ n \end{pmatrix}\begin{pmatrix} t \\ k \end{pmatrix}((m+n+k)!\,|y_{mnk}|)^{\frac{1}{m+n+k}} = 0.$

Thus, we have that $x \in P_{\chi^3}$. In addition note that

$d(x,y)_{P_{\chi^3}} = sup_{r,s,t}\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}\begin{pmatrix} r \\ m \end{pmatrix}\begin{pmatrix} s \\ n \end{pmatrix}\begin{pmatrix} t \\ k \end{pmatrix}\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}$

$(-1)^{(r-m)+(s-n)+(t-k)}\begin{pmatrix} r \\ m \end{pmatrix}\begin{pmatrix} s \\ n \end{pmatrix}\begin{pmatrix} t \\ k \end{pmatrix}((m+n+k)!\,|x_{mnk}-y_{mnk}|)^{\frac{1}{m+n+k}}$

$= sup_{r,s,t}\left((r+s+t)!\left|x_{rst}-y_{rst}\right|\right)^{\frac{1}{r+s+t}}$
$= d(x,y)_{\chi^3} < \infty.$

Consequently , T is surjective and is metric preserving. Hence, T is a linear bijection which therefore says us that the spaces P_{χ^3} to χ^3 are linearly isomorphic. In the same way, it can be shown that P_{Λ^3} are linearly isomorphic to Λ^3, respectively and so we omit the detail.

B. Theorem

Let $(m,n,k)\in\mathbb{N}^3$ be a fixed triple number and $b^{(mnk)}=\left\{b_{rst}^{(mnk)}\right\}_{(r,s,t)\in\mathbb{N}^3}$, where

$$b_{rst}^{(mnk)} = \begin{cases} 0, if\ 0\le r\le m, s\le n, t\le k \\ (-1)^{(r-m)+(s-n)+(t-k)}\begin{pmatrix} r \\ m\end{pmatrix}\begin{pmatrix} s \\ n\end{pmatrix}\begin{pmatrix} t \\ k\end{pmatrix} & \text{if } r\ge m, s\ge n, t\ge k. \end{cases}$$

Then the following assertions are true:

(i) The triple sequence $\left\{b_{rst}^{(mnk)}\right\}$ is a basis for the space P_{χ^3} and every $x\in P_{\chi^3}$ has a unique representation of the form $\mu=\sum_m\sum_n\sum_k\lambda_{mnk}b^{(mnk)}$, where $\lambda_{mnk}=\left(P\left((m+n+k)!\left|x_{mnk}\right|\right)^{\frac{1}{m+n+k}}\right)_{m,n,k}$ for all $m,n,k\in\mathbb{N}$.

C. Proposition

The triple sequence P_{χ^3} is a linear set over the set of complex numbers $\mathbb{C}$.

Proof: It is trivial. Therefore, the proof is omitted.

D. Proposition

$\left(P_{\chi^3}\right)^{\delta} \overset{\subset}{\neq} P_{\Lambda^3}$

Proof: Let $\gamma\in\delta-$ dual of P_{χ^3}. Then $|\mu_{mnk}\gamma_{mnk}|\le M^{m+n+k}$ for some constant $M>0$ and for each $\mu\in P_{\chi^3}$. Therefore, $|\gamma_{mnk}|\le M^{m+n+k}$ for each m,n,k by taking $\mu=(\Im_{mnk})$. This implies that $\gamma\in P_{\Lambda^3}$. Thus,

$$\left(P_{\chi^3}\right)^{\delta}\subset P_{\Lambda^3} \tag{9}$$

we now choose the triple sequences (γ_{mnk}) and (μ_{mnk}) by $(\gamma_{mnk})=1$ for all m,n and k, and by

$$(m+2)!x_{m11}=\left[2\tfrac{r!}{m!(r-m)!}\tfrac{s!}{1!(s-1)!}\tfrac{t!}{1!(t-k)!}\right]^{(m+2)^2} \text{ and}$$
$$(m+n+k)!x_{mnk}=\left[2\tfrac{r(r-1)!}{m(m-1)!(r-m)!}\tfrac{s(s-1)!}{n(n-1)!(s-n)!}\tfrac{t(t-1)!}{k(k-1)!(t-k)!}\right]=0(r,s,t=0) \text{ for all } m,n,k=0$$

Obviously, $\gamma\in P_{\Lambda^3}$ and since $(m+n+k)!x_{mnk}=0$ for all $m,n,k=0$, $(m+n+k)!(x_{mnk})$ converges to zero. Hence, $\mu\in P_{\chi^3}$. But

$$\left((m+2)!\left|a_{m11}x_{m11}\right|\right)^{\frac{1}{m+n+k}}=\left[2\tfrac{r!}{m!(r-m)!}\tfrac{s!}{1!(s-1)!}\tfrac{t!}{1!(t-1)!}\right]^{(m+2)^2}\to\infty \text{ as } m\to\infty, \text{ hence}$$

$$\mu \notin \left(P_{\chi^3}\right)^{\delta} \tag{10}$$

From (3.1) and (3.2), we are granted $\left(P_{\chi^3}\right)^{\delta} \overset{\subset}{\neq} P_{\chi^3}$. This completes the proof.

E. Proposition:

The dual space of P_{χ^3} is P_{χ^3}.In other words $(P_{\chi^3})^* = P_{\chi^3}$.

Proof: We recall that $\Im_{mnk} = \begin{pmatrix} 0 & 0 & ...0 & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \\ . \\ . \\ . \\ 0 & 0 & ...\frac{1}{(m+n+k)!} & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \end{pmatrix}$

with $\frac{1}{(m+n+k)!}$ in the $(m,n,k)th$ position and zero's else where, with
$\mu = \Im_{mnk}$,

which is a triple Pascal chi sequence. Hence, $\Im_{mnk} \in P_{\chi^3}; f(x) = \sum_{m,n,k=1}^{\infty} \mu_{mnk}\gamma_{mnk}$ with $\mu \in P_{\chi^3}$ and $f \in (P_{\chi^3})^*$, where $(P_{\chi^3})^*$ is the dual space of P_{χ^3}. Take $\mu = (\mu_{mnk}) = \Im_{mnk} \in P_{\chi^3}$.Then,

$$|\gamma_{mnk}| \leq \|f\| \, d(\Im_{mnk}, 0) < \infty \qquad \forall m,n,k. \tag{11}$$

Thus, (γ_{mnk}) is a bounded sequence and hence an triple Pascal analytic sequence. In other words, $\gamma \in P_{\Lambda^3}$. Therefore $(P_{\chi^3})^* = P_{\Lambda^3}$.This completes the proof.

F. Proposition:

$(P_{\Lambda^3})^{\beta} \overset{\subset}{\neq} P_{\chi^3}$

Proof: Step 1: Let $(\mu_{mnk}) \in (P_{\Lambda^3})^{\beta}$,

$$\sum_{m,n,k=1}^{\infty} |\mu_{mnk}\gamma_{mnk}| < \infty \forall (\gamma_{mnk}) \in P_{\Lambda^3} \tag{12}$$

Let us assume that $(\mu_{mnk}) \notin P_{\chi^3}$. Then, there exists a strictly increasing sequence of positive integers $(m_p + n_p + k_p)$ such that

$$(m_p + n_p + k_p)! \left| x_{(m_p+n_p+k_p)} \right| > \frac{1}{\left[2 \frac{r!}{m!(r-m)!} \frac{s!}{1!(s-1)!} \frac{t!}{1!(t-k)!} \right]^{(m_p+n_p+k_p)}}, (p = 1, 2, 3, \dots) \tag{13}$$

Let

$$(m_p+n_p+k_p)!\gamma_{\left(m_p+n_p+k_p\right)} = \left[2\frac{r!}{m!(r-m)!}\frac{s!}{1!(s-1)!}\frac{t!}{1!(t-k)!}\right]^{\left(m_p+n_p+k_p\right)} \text{ for } (p=1,2,3,\ldots)$$

$$\gamma_{mnk} = 0 \text{ otherwise}$$

Then, $(\gamma_{mnk}) \in P_{\Lambda^3}$. However,
$\sum_{m,n,k=1}^{\infty} |\mu_{mnk}\gamma_{mnk}| = \sum_{p=1}^{\infty}(m_p+n_p+k_p)!\left|\mu_{\left(m_pn_pk_p\right)}\gamma_{\left(m_pn_pk_p\right)}\right| > 1+1\ldots\cdot$
We know that the infinite series 1+1+1+... diverges. Hence $\sum_{m,n,k=1}^{\infty}|\mu_{mnk}\gamma_{mnk}|$ diverges. This contradicts (3.4). Hence $(\mu_{mnk}) \in P_{\chi^3}$. Therefore,

$$(P_{\Lambda^3})^{\beta} \subset P_{\chi^3} \tag{14}$$

and $\gamma_{mnk} = \mu_{mnk} = 0\,(m>1)$ for all n,k then obviously $\mu \in P_{\chi^3}$ and $\gamma \in P_{\Lambda^3}$, but $\sum_{m,n,k=1}^{\infty}\mu_{mnk}\gamma_{mnk} = \infty$. Hence,

$$\gamma \notin (P_{\Lambda^3})^{\beta} \tag{15}$$

From (3.6) and (3.7), we are granted $(P_{\Lambda^3})^{\beta} \overset{\subset}{\neq} P_{\chi^3}$. This completes the proof.

G. Definition

Let $p = (p_{mnk})$ be a triple sequence of positive real numbers. Then,

$$P_{\chi^3}(p) = \left(lim_{rst\to\infty}\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}\begin{pmatrix} r \\ m \end{pmatrix}\begin{pmatrix} s \\ n \end{pmatrix}\begin{pmatrix} t \\ k \end{pmatrix}\left((m+n+k)!\,|x_{mnk}|\right)^{\frac{1}{m+n+k}} = 0\right)^{p_{mnk}}. \tag{16}$$

Suppose that p_{mnk} is a constant for all m,n,k then $P_{\chi^3}(p) = P_{\chi^3}$.

H. Proposition

Let $0 \le p_{mnk} \le q_{mnk}$ for all $m,n,k \in \mathbb{N}$ and let $\left\{\frac{q_{mnk}}{p_{mnk}}\right\}$ be bounded. Then $P_{\chi^3}(q) \subset P_{\chi^3}(p)$.

Proof: Let

$$\mu \in P_{\chi^3}(q), then \tag{17}$$

$$\left(lim_{r,s,t\to\infty}\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}\begin{pmatrix} r \\ m \end{pmatrix}\begin{pmatrix} s \\ n \end{pmatrix}\begin{pmatrix} t \\ k \end{pmatrix}\left((m+n+k)!\,|x_{mnk}|\right)^{\frac{1}{m+n+k}} = 0\right)^{q_{mnk}}. \tag{18}$$

Let $t_{mnk} = \left(\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}\begin{pmatrix} r \\ m \end{pmatrix}\begin{pmatrix} s \\ n \end{pmatrix}\begin{pmatrix} t \\ k \end{pmatrix}\left((m+n+k)!\,|x_{mnk}|\right)^{\frac{1}{m+n+k}}\right)^{q_{mnk}}$, and let $\alpha_{mnk} = p_{mnk}/q_{mnk}$. Since $p_{mnk} \le q_{mnk}$, we have $0 \le \alpha_{mnk} \le 1$. Let $0 < \alpha < \alpha_{mnk}$. then

$$u_{mnk} = \begin{cases} t_{mnk}, & \text{if } (t_{mnk} \ge 1) \\ 0, & \text{if } (t_{mnk} < 1) \end{cases}$$

$$v_{mnk}=\begin{cases} 0, & \text{if } (t_{mnk}\geq 1)\\ t_{mnk}, & \text{if } (t_{mnk}<1)\end{cases} \tag{19}$$

$t_{mnk}=u_{mnk}+v_{mnk},\, t_{mnk}^{\alpha_{mnk}}=u_{mnk}^{\alpha_{mnk}}+v_{mnk}^{\alpha_{mnk}}.$

Now, it follows that

$$u_{mnk}^{\alpha_{mnk}}\leq u_{mnk}\leq t_{mnk},\, v_{mnk}^{\alpha_{mnk}}\leq u_{mnk}^{\alpha} \tag{20}$$

Since $t_{mnk}^{\alpha_{mnk}}=u_{mnk}^{\alpha_{mnk}}+v_{mnk}^{\alpha_{mnk}},\, we\, have\, t_{mnk}^{\alpha_{mnk}}\leq t_{mnk}+v_{mnk}^{\alpha}$. Thus,

$$\left(\left(\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}\begin{pmatrix} r\\ m\end{pmatrix}\begin{pmatrix} s\\ n\end{pmatrix}\begin{pmatrix} t\\ k\end{pmatrix}\left((m+n+k)!\,|x_{mnk}|\right)^{\frac{1}{m+n+k}}\right)^{q_{mnk}}\right)^{\alpha_{mnk}}\leq$$
$$\left(\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}\begin{pmatrix} r\\ m\end{pmatrix}\begin{pmatrix} s\\ n\end{pmatrix}\begin{pmatrix} t\\ k\end{pmatrix}\left((m+n+k)!\,|x_{mnk}|\right)^{\frac{1}{m+n+k}}\right)^{q_{mnk}}$$
$$\left(\left(\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}\begin{pmatrix} r\\ m\end{pmatrix}\begin{pmatrix} s\\ n\end{pmatrix}\begin{pmatrix} t\\ k\end{pmatrix}\left((m+n+k)!\,|x_{mnk}|\right)^{\frac{1}{m+n+k}}\right)^{q_{mnk}}\right)^{\left(p_{mnk}/q_{mnk}\right)}\leq$$
$$\left(\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}\begin{pmatrix} r\\ m\end{pmatrix}\begin{pmatrix} s\\ n\end{pmatrix}\begin{pmatrix} t\\ k\end{pmatrix}\left((m+n+k)!\,|x_{mnk}|\right)^{\frac{1}{m+n+k}}\right)^{q_{mnk}}\cdots\cdots *$$

which yields

$$\left(\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}\begin{pmatrix} r\\ m\end{pmatrix}\begin{pmatrix} s\\ n\end{pmatrix}\begin{pmatrix} t\\ k\end{pmatrix}\left((m+n+k)!\,|x_{mnk}|\right)^{\frac{1}{m+n+k}}\right)^{p_{mnk}}$$
$$\leq\left(\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}\begin{pmatrix} r\\ m\end{pmatrix}\begin{pmatrix} s\\ n\end{pmatrix}\begin{pmatrix} t\\ k\end{pmatrix}\left((m+n+k)!\,|x_{mnk}|\right)^{\frac{1}{m+n+k}}\right)^{q_{mnk}}.$$

However, $\left(lim_{rst\to\infty}\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}\begin{pmatrix} r\\ m\end{pmatrix}\begin{pmatrix} s\\ n\end{pmatrix}\begin{pmatrix} t\\ k\end{pmatrix}\left((m+n+k)!\,|x_{mnk}|\right)^{\frac{1}{m+n+k}}=0\right)^{q_{mnk}}$ (by(3.10)). Thus,

$$\left(lim_{rst\to\infty}\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}\begin{pmatrix} r\\ m\end{pmatrix}\begin{pmatrix} s\\ n\end{pmatrix}\begin{pmatrix} t\\ k\end{pmatrix}\left((m+n+k)!\,|x_{mnk}|\right)^{\frac{1}{m+n+k}}=0\right)^{p_{mnk}}.$$

Hence,

$$\mu\in P_{\chi^3}(p). \tag{21}$$

Hence (3.9) and (3.13), we are granted

$$P_{\chi^3}(q)\subset P_{\chi^3}(p). \tag{22}$$

This completes the proof.

I. Proposition

(a) Let $0 < inf p_{mnk} \leq p_{mnk} \leq 1$, then $P_{\chi^3}(p) \subset P_{\chi^3}$.

(b)If $1 \leq p_{mnk} \leq supp_{mnk} < \infty$, then $P_{\chi^3} \subset P_{\chi^3}(p)$.

Proof: The above statements are special cases of proposition 3.8. Therefore, it can be proved by similar arguments.

J. Proposition

If $0 < p_{mnk} \leq q_{mnk} < \infty$ for each m,n,k then $P_{\chi^3}(p) \subseteq P_{\chi^3}(q)$.

Proof: Let $\mu \in P_{\chi^3}(p)$, then

$$\left(lim_{r,s,t\to\infty} \sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t} \begin{pmatrix} r \\ m \end{pmatrix} \begin{pmatrix} s \\ n \end{pmatrix} \begin{pmatrix} t \\ k \end{pmatrix} \left((m+n+k)!\,|x_{mnk}|\right)^{\frac{1}{m+n+k}} = 0 \right)^{p_{mnk}} \tag{23}$$

$\left(\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t} \begin{pmatrix} r \\ m \end{pmatrix} \begin{pmatrix} s \\ n \end{pmatrix} \begin{pmatrix} t \\ k \end{pmatrix} \left((m+n+k)!\,|x_{mnk}|\right)^{\frac{1}{m+n+k}} \right)^{q_{mnk}} \leq$
$\left(\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t} \begin{pmatrix} r \\ m \end{pmatrix} \begin{pmatrix} s \\ n \end{pmatrix} \begin{pmatrix} t \\ k \end{pmatrix} \left((m+n+k)!\,|x_{mnk}|\right)^{\frac{1}{m+n+k}} \right)^{p_{mnk}} \cdots\cdots *$, then
$\left(lim_{rst\to\infty} \sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t} \begin{pmatrix} r \\ m \end{pmatrix} \begin{pmatrix} s \\ n \end{pmatrix} \begin{pmatrix} t \\ k \end{pmatrix} \left((m+n+k)!\,|x_{mnk}|\right)^{\frac{1}{m+n+k}} = 0 \right)^{q_{mnk}}$ (by using (3.15)). We have $\mu \in P_{\chi^3}(q)$. Hence, $P_{\chi^3}(p) \subseteq P_{\chi^3}(q)$. This completes the proof.

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