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A Class of Almost Contact Metric Manifolds

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The object of the present paper is to study the class of almost contact metric manifolds and also investigated some properties of this class.

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I. INTRODUCTION

In 1959, an odd-dimensional manifold M^{2n+1}, Gray [4] defined an almost contact structure as a structural group to $U(n)\times 1$. The structure tensor M^{2n+1}, has an almost contact structure or sometimes (ϕ,ξ,η)-structure if it admits a tensor field ϕ of type $(1,1)$, a vector field ξ, and a 1-form η satisfying

$$\phi^2 = -I + \eta\otimes\xi,\ \ \eta(\xi) = 1, \tag{1}$$

$$\phi(\xi) = 0,\ \ \eta\circ\phi = 0. \tag{2}$$

If a manifold M^{2n+1} with a (ϕ,ξ,η)-structure admits a Riemannian metric g such that

$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{3}$$

then M^{2n+1} has an almost contact metric structure and g is called a compatible metric. Setting $Y=\xi$ in (3), we conclude that

$$\eta(X) = g(X,\xi). \tag{4}$$

In 1972, a special almost contact metric structure introduced by Kenmotsu [7] seems to play a role here. An almost contact metric manifold (M,ϕ,ξ,η,g) is called a Kenmotsu manifold if it satisfies

$$(\nabla_X\phi)Y = g(\phi X,Y)\xi - \eta(Y)\phi X, \tag{5}$$

where ∇ denotes the Riemannian connection or Levi-Civita connection of g. Kenmotsu gave a local characterization of this structure.

In 1980, the classification of Gray and Hervella [5] of almost Hermitian manifolds there appears a class $\mathscr{W}_4$, of Hermitian manifolds which are closely related to locally conformally K$\ddot{a}$hler manifolds. If we consider $M_1\times\mathbb{R}$ with the almost complex structure

$$J(X,f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt}),$$

where f is a real valued function , is integrable then the structure is said to be Sasakian.

In 1985, Oubina [11] introduced the notion of a trans-Sasakian structure as an almost contact metric structure (ϕ,ξ,η,g) for which the almost Hermitian manifold $(M_1\times\mathbb{R},J,G)$ belongs to the class $\mathscr{W}_4$, where G denotes the product metric. This may be expressed by the condition

$$(\nabla_X\phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X,Y)\xi - \eta(Y)\phi X),$$

for function α and β on M and the trans-Sasakian structure is said to be of type (α,β). If β but not α (α but not β) vanishes, the structure is α-Sasakian (resp.β-Sasakian) [6]. In

1992, Marrero [9] showed that a trans-Sasakian manifold of dimension ≥ 5 is either α-Sasakian, β-Sasakian or cosymplectic.

An almost contact metric structure (ϕ,ξ,η,g) on M^{2n+1} is called a contact metric structure (ϕ,ξ,η,g) if it satisfies [2]

$$d\eta(X,Y)=\frac{1}{2}[(\nabla_X\eta)(Y)-(\nabla_Y\eta)(X)]=g(X,\phi Y)=\Phi(X,Y), \tag{6}$$

where Φ denotes the fundamental 2-form of the almost contact metric structure. A manifold M^{2n+1} with a contact metric structure (ϕ,ξ,η,g) is said to be a contact metric manifold. Moreover, $\nabla_X\xi=-\phi X-\phi hX$, $\nabla_\xi\xi=0$, where h denotes a $(1,1)$ type tensor h by $h=\frac{1}{2}\pounds_\xi\phi$ ($\pounds$ means the Lie differentiation). A contact metric manifold $(M^{2n+1},\phi,\xi,\eta,g)$ for which ξ is a killing vector is called a K-contact metric manifold. It is well-known that a contact metric manifold is K-contact if and only if $h=0$.

In this paper we study the special type of an almost contact metric structure. The paper is organized as follows: After introduction in section 2, from the definition by means of the tensor equations it is easily verified that the structure is a contact metric manifold, but ξ is not a killing vector field and we construct an example to verify this special type of the almost contact metric structure. The curvature tensor, the Ricci tensor and some properties of this structure with an example have been studied in Section 3. Section 4 is devoted to a study of η-Einstein manifolds. Finally, we have discussed the Concircular curvature tensor on this almost contact metric structure.

II. DEFINITION AND EXAMPLE

Let us consider a class of almost contact metric manifolds which satisfy the following conditions

$$\begin{aligned}(\nabla_X\phi)(Y)=&-\eta(Y)X-\eta(Y)hX-\eta(Y)\phi X\\&+g(X,Y)\xi+g(hX,Y)\xi+g(\phi X,Y)\xi,\end{aligned} \tag{7}$$

$$\nabla_X\xi=X-\phi X-\phi hX-\eta(X)\xi. \tag{8}$$

In 1969, Eum [3] studied the integrability of invariant hypersurfaces immersed in an almost contact metric manifold which satisfies

$$g((\nabla_X\phi)Y,Z)=(\nabla_X\eta)(\eta(Y)\phi Z-\eta(Z)\phi Y). \tag{9}$$

If we assume (8) on the almost contact metric manifold, then (7) is equivalent to (9) and we write that

$$(\nabla_X\eta)Y=g(X,Y)-g(\phi X,Y)-g(\phi hX,Y)-\eta(X)\eta(Y). \tag{10}$$

Combining (6), (10), we conclude that the class of almost contact metric manifolds which satisfy the equations (7) and (8) is a class of the contact metric manifolds. The author has been preferred the name of this contact metric manifold as a Barman manifold.

Taking the Lie differentiation of g with respect to ξ and using (3) and (8), we see that

$$(\pounds_\xi g)(X,Y)=2g(X,\phi Y)\neq 0.$$

Therefore, ξ is not a killing vector field.

Summing up all of the above discussion we can state the following proposition:

Proposition II.1. *If a class of almost contact metric manifolds which the M^{2n+1}, which satisfy the condition $(\nabla_X\phi)(Y)=-\eta(Y)X-\eta(Y)hX-\eta(Y)\phi X+g(X,Y)\xi+g(hX,Y)\xi+g(\phi X,Y)\xi$ and $\nabla_X\xi=X-\phi X-\phi hX-\eta(X)\xi$, then*
(i) M^{2n+1} is the integrability of invariant hypersurfaces immersed due to Eum [3],
(ii) $(\nabla_X\eta)Y=g(X,Y)-g(\phi X,Y)-g(\phi hX,Y)-\eta(X)\eta(Y)$,
(iii) ξ is not a killing vector field on M^{2n+1}.

Taking the covariant derivative of h and ϕh, we get

Proposition II.2. *Under the same assumption as Proposition II.1*

$$(\nabla_X h)Y = -\eta(Y)hX + \eta(Y)h\phi X - \eta(Y)h^2\phi X \\ -g(X,\phi hY)\xi - g(X,hY)\xi - g(h^2X,Y)\xi. \quad (11)$$

$$(\nabla_X \phi h)Y = -\eta(Y)\phi hX + \eta(Y)hX + \eta(Y)h^2X \\ +g(X,hY)\xi + g(\phi X,hY)\xi + g(h^2X,Y)\xi. \quad (12)$$

Since the proof of Proposition III.2 follows by a routine calculation, we shall omit it.

Example II.1. *We begin with a result of Tashiro [13] that every C^∞ orientable hypersurface of an almost complex manifold has an almost contact structure.*

Let $(\tilde{M}^{2n+2},J)$ be an almost complex manifold and $\iota : M^{2n+1} \longrightarrow \tilde{M}^{2n+2}$ a C^∞ orientable hypersurface. There exists a transverse vector field ν along M^{2n+1} such that $J\nu$ is tangent. For if $J\iota_* X$ is tangent for every tangent vector X, $J\iota_* X = \iota_* fX$ defines a $(1,1)$-tensor field f on M^{2n+1}. Applying J, we have $f^2 = -I$ on M^{2n+1}, making M^{2n+1} an almost complex manifold, a contradiction. Thus there exists a vector field ξ on M^{2n+1} such that $\nu = J\iota_*\xi$ is transverse.

Define a tensor field ϕ of type (1, 1) and a 1-form η on M^{2n+1} by

$$J\iota_* X = \iota_*\phi X + \eta(X)\nu; \quad (13)$$

then applying J, we have

$$-\iota_* X = \iota_*\phi^2 X + \eta(\phi X)\nu - \eta(X)\iota_*\xi$$

and hence $\phi^2 = -I + \eta\otimes\xi$ and $\eta\circ\phi = 0$. Taking $X = \xi$ in equation (13) *gives $\nu = \iota_*\phi\xi + \eta(\xi)\nu$ and hence $\phi\xi = 0$ and $\eta(\xi) = 1$. Therefore (ϕ,η,ξ) is an almost contact structure on M^{2n+1}.*

The Gauss-Weingarten equations, we obtain

$$(\tilde{\nabla}_X J)\xi = \tilde{\nabla}_X \nu - J(\nabla_X\xi + \sigma(X,\xi)) = 0,$$

where $\sigma(X,Y) = -\frac{1}{r}g(X,Y)\nu$ denotes the second fundamental form and r be a sphere of radius.
The above equation implies that

$$0 = (\tilde{\nabla}_X J)\xi = \frac{1}{r}X - \phi\nabla_X\xi - \frac{1}{r}\eta(X)\xi + \frac{1}{r}\phi X + \frac{1}{r}hX.$$

Applying ϕ, we have $\nabla_X\xi = \frac{1}{r}(X - \phi X - \phi hX - \eta(X)\xi)$. This in turn yields

$$d\eta(X,Y) = \frac{1}{2}(g(\nabla_X\xi,Y) - g(\nabla_Y\xi,X)) = \frac{1}{r}g(X,\phi Y).$$

Thus for $r \neq 1$, g is not an associated metric, but this situation is easily rectified. The structure $\bar{\eta} = \frac{1}{r}\eta$, $\bar{\xi} = r\xi$, $\bar{\eta} = \frac{1}{r}\eta$, $\bar{\phi} = \phi$, $\bar{h} = h$ and $\bar{g} = \frac{1}{r^2}g$ is the new contact metric structure (Barman manifold).

III. CURVATURE TENSOR AND RICCI TENSOR

Analogous to the definitions of the curvature tensor R of M with respect to the Levi-Civita connection ∇,

$$R(X,Y)\xi = \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{[X,Y]}\xi, \quad (14)$$

where $X,Y \in \chi(M)$, the set of all differentiable vector fields on M.

Now using (4), (7), (8), (10) and (12) in (14), we obtain

$$R(X,Y)\xi = \eta(Y)X + \eta(Y)\phi X - \eta(Y)h\phi X - \eta(Y)h^2X \\ -\eta(X)Y - \eta(X)\phi Y + \eta(X)h\phi Y + \eta(X)h^2Y. \quad (15)$$

We state the following proposition of Blair [2] which will be used to determined the value of h^2 :

Proposition III.1. *[2] On a contact metric manifold M^{2n+1}, we have the following formulas:*

$$(\nabla_\xi h)(X) = \phi X - h^2\phi X - \phi R(X,\xi)\xi$$

and

$$\frac{1}{2}[(R(\xi,X)\xi - \phi(R(\xi,\phi X)\xi] = h^2X + \phi^2X.$$

Combining (1), (2), (12), (15) and Proposition III.1, we decided that

$$h^2X = X - \eta(X)\xi,\ g(hX,hY) = g(X,Y) - \eta(X)\eta(Y). \quad (16)$$

From (15) and (16), we derived that

$$R(X,Y)\xi = \eta(Y)\phi X - \eta(Y)h\phi X - \eta(X)\phi Y + \eta(X)h\phi Y. \quad (17)$$

Putting $X = \xi$ in (17), we see that

$$R(\xi,Y)\xi = -\phi Y + h\phi Y. \quad (18)$$

In view of (17), implies that

$$\eta(R(X,Y)Z) = \eta(Y)g(h\phi X,Z) - \eta(Y)g(\phi X,Z) + \eta(X)g(\phi Y,Z) - \eta(X)g(h\phi Y,Z).$$

Combining (1), (2), (3) and (4), we have decided that

$$g(\phi X,Y) = -g(X,\phi Y).$$

Putting $X = Y = e_i$; $i = 1,2,3,...,2n+1$ in the above equation, then we conclude that

$$g(\phi e_i,e_i) = 0. \quad (19)$$

Corollary III.1. *[2] On a contact metric manifold M^{2n+1}, the relation $d\eta = 0$ holds.*

By the Corollary III.1, we see

$$0 = d\eta = -div\xi = -\Sigma_{i=1}^{2n+1} g(\nabla_{e_i}\xi,e_i). \quad (20)$$

From the equations (8) and (20), we get

$$\Sigma_{i=1}^{2n+1} g(e_i,e_i) - \Sigma_{i=1}^{2n+1} g(\phi e_i,e_i) - \Sigma_{i=1}^{2n+1} g(\phi he_i,e_i) - \Sigma_{i=1}^{2n+1} \eta(e_i)g(e_i,\xi) = 0. \quad (21)$$

In virtue of (19) and (21), it implies that

$$\alpha = g(\phi he_i,e_i) = 2n. \quad (22)$$

The above discussion help us to state the following theorem:

Theorem III.1. *Any contact metric manifold $\alpha = g(\phi he_i,e_i)$ is equal to zero, but the new contact metric manifolds (Barman manifolds) $\alpha = g(\phi he_i,e_i)$ is equal to $2n$.*

Now contraction X from (17) and using (22), it can be easily seen that

$$S(X,\xi) = 2n\eta(X), \quad (23)$$

where S be the Ricci tensor of M.

Therefore, considering all the cases we can state the following proposition:

Proposition III.2. *Under the same assumption as Proposition II.1*
(i) $R(X,Y)\xi = \eta(Y)\phi X - \eta(Y)h\phi X - \eta(X)\phi Y + \eta(X)h\phi Y$,
(ii) $R(\xi,Y)\xi = -\phi Y + h\phi Y$,
(iii) $\eta(R(X,Y)Z) = \eta(Y)g(h\phi X,Z) - \eta(Y)g(\phi X,Z) + \eta(X)g(\phi Y,Z) - \eta(X)g(h\phi Y,Z)$,
(iv) $S(X,\xi) = 2n\eta(X)$.

Example III.1. *We consider the 5-dimensional manifold $M = \{(x,y,z,u,v) \in R^5\}$, where (x,y,z,u,v) are the standard coordinate in R^5.*

We choose the vector fields

$$e_1 = \frac{\partial}{\partial x},\ e_2 = \frac{\partial}{\partial y},\ e_3 = e^{-v}\frac{\partial}{\partial z},\ e_4 = e^{-v}\frac{\partial}{\partial u},\ e_5 = \frac{\partial}{\partial v},$$

which are linearly independent at each point of M.

Let g be the Riemannian metric defined by

$$g(e_i,e_j) = 0,\ i \neq j,\ i,j = 1,2,3,4,5$$

and

$$g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_3) = g(e_4,e_4) = g(e_5,e_5) = 1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z,e_5)$, for any $Z \in M^{2n+1}$.

Let ϕ and h are tensor field defined by

$$\phi e_1 = e_2,\ \phi e_2 = -e_1,\ \phi e_3 = e_4,\ \phi e_4 = -e_3,\ \phi e_5 = 0$$

and

$$he_1 = -e_1,\ he_2 = -e_2,\ he_3 = -e_3,\ he_4 = -e_4,\ he_5 = 0.$$

Using the linearity of ϕ, h and g, we have $\eta(e_5) = 1$, $\phi^2(Z) = -Z + \eta(Z)e_5$, $g(\phi Z, \phi U) = g(Z,U) + \eta(Z)\eta(U)$, $h^2 Z = Z - \eta(Z)e_5$ and $g(hZ,hU) = g(Z,U) - \eta(Z)\eta(U)$, for any $U,Z \in M^{2n+1}$. Thus, for $e_5 = \xi$, $M(\phi,\xi,\eta,g)$ defines an almost contact metric manifold.

Then we have

$$[e_1,e_2] = [e_1,e_3] = [e_1,e_4] = [e_1,e_5] = [e_2,e_3] = 0,$$

$$[e_4,e_5] = e_4,\ [e_2,e_4] = [e_2,e_5] = [e_3,e_4] = 0,\ [e_3,e_5] = e_3.$$

The Riemannian connection ∇ of the metric tensor g is given by Koszul's formula which is given by

$$2g(\nabla_X Y,Z) = Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) - g(X,[Y,Z]) + g(Y,[X,Z]) + g(Z,[X,Y]).$$

Taking $e_5 = \xi$ and using Koszul's formula we get the following

$$\nabla_{e_1} e_1 = 0,\ \nabla_{e_1} e_2 = 0,\ \nabla_{e_1} e_3 = 0,\ \nabla_{e_1} e_4 = 0,\ \nabla_{e_1} e_5 = 0,$$

$$\nabla_{e_2} e_1 = 0,\ \nabla_{e_2} e_2 = 0,\ \nabla_{e_2} e_3 = 0,\ \nabla_{e_2} e_4 = 0,\ \nabla_{e_2} e_5 = 0,$$

$$\nabla_{e_3} e_1 = 0,\ \nabla_{e_3} e_2 = 0,\ \nabla_{e_3} e_3 = 0,\ \nabla_{e_3} e_4 = 0,\ \nabla_{e_3} e_5 = e_3,$$

$$\nabla_{e_4} e_1 = 0,\ \nabla_{e_4} e_2 = 0,\ \nabla_{e_4} e_3 = 0,\ \nabla_{e_4} e_4 = 0,\ \nabla_{e_4} e_5 = e_4,$$

$$\nabla_{e_5} e_1 = 0,\ \nabla_{e_5} e_2 = 0,\ \nabla_{e_5} e_3 = -e_3,\ \nabla_{e_5} e_4 = -e_4,\ \nabla_{e_5} e_5 = 0.$$

In view of the above relations, we see that $\nabla_X \xi = X - \phi X - \phi hX - \eta(X)\xi$ and $(\nabla_X \phi)(Y) = -\eta(Y)X - \eta(Y)hX - \eta(Y)\phi X + g(X,Y)\xi + g(hX,Y)\xi + g(\phi X,Y)\xi$, for all $e_5 = \xi$. Therefore the manifold is another special type almost contact metric manifold (Barman manifold) with the structure (ϕ,ξ,η,g).

IV. η-EINSTEIN MANIFOLD

Definition IV.1. *In an almost contact metric manifold, if the Ricci tensor (S) satisfies*

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{24}$$

where a and b are scalar functions, then the manifold is called an η-Einstein manifold.

Putting $Y = \xi$ in (24) and using (1) and (4), we derived that

$$S(X,\xi) = (a+b)\eta(X). \tag{25}$$

Making from (23) and (25), we calculate that

$$a + b = 2n.$$

This leads to the following theorem:

Theorem IV.1. *Any contact metric manifold M^{2n+1} is an η-Einstein manifold, for $a+b = 2n$. Similarly, if the new contact metric manifold (Barman manifold) under the assumption as the (7) and (8) is also η-Einstein manifold,then we get $a+b = 2n$.*

V. CONCIRCULAR CURVATURE TENSOR

A transformation of an $2n+1$-dimensional Riemannian manifold M, which transforms every geodesic circle of M into a geodesic circle, is called a concircular transformation ([8], [14]). A concircular transformation is always a conformal transformation [8]. Here geodesic circle means a curve in M whose first curvature is constant and whose second curvature is identically zero. Thus the geometry of concircular transformations, i.e., the concircular geometry, is a generalization of inversive geometry in the sense that the change of metric is more general than that induced by a circle preserving diffeomorphism (see also [1]). An interesing invariant of a concircular transformation is the concircular curvature tensor $\mathbb{W}$. It is defined by ([14], [15])

$$\mathbb{W}(X,Y)Z = R(X,Y)Z - \frac{r}{2n(2n+1)}[g(Y,Z)X - g(X,Z)Y]. \quad (26)$$

where X, Y, Z on M and $\mathbb{W}$ is the concircular curvature tensor and r is the scalar curvature respectively. Riemannian manifolds with vanishing concircular curvature tensor are of constant curvature. Thus the concircular curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature.

Definition V.1. *A new contact metric manifolds (Barman manifolds) is said to be ξ-concircularly flat if $\mathbb{W}(X,Y)\xi = 0$, where $X,Y \in \chi(M)$.*

Putting $Z = \xi$ in (26) and using (4) and (23), we have

$$\mathbb{W}(X,Y)\xi = R(X,Y)\xi - \frac{r}{4n^2(2n+1)}[S(Y,\xi)X - S(X,\xi)Y]. \quad (27)$$

If $R(X,Y)\xi = 0$, then $\mathbb{W}(X,Y)\xi = 0$.

In view of above discussions we can state the following theorem:

Theorem V.1. *A new contact metric manifold (Barman manifold) is ξ-concircularly flat if $R(X,Y)\xi$ vanishes.*

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